Study on Nonlinear Dynamic Characteristics of 2K-H Planetary Gear System

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ABSTRACT

A 2K-H type planetary gear reducer with involute spur gear is adopted as the research object. Considering the time-varying meshing stiffness, the time-varying backlash of the gear meshing and the time varying pressure angle and other nonlinear factors, the lateral-torsional-swing coupled nonlinear dynamic model of the planetary gear transmission system is established. According to the equations of Lagrange, the dynamic differential equations are derived. By introducing the state variable, the second-order differential equation of the coupled system is reduced to the first-order differential equation. The fifth-order step-varying adaptive Runge-Kutta numerical integration method is used to solve the dynamic equation. Compared with the previous dynamic models, this nonlinear dynamic model for the planetary gear transmission system is more closely to the practical operating conditions of the transmission system, which has laid a foundation for further evaluating the dynamic characteristics of the planetary gear transmission system in the future.

INTRODUCTION

The planetary gear system is not only widely applied to the automotive and vessels field, but also more and more popular among the CNC machine tools due to its excellent properties including compact structure, large transmission power and high bearing capacity[1]. A large number of meaningful researches on the dynamic characteristics of the planetary gear transmission system are carried out by many domestic and foreign scholars. Considering the meshing stiffness fluctuation and the meshing frequency errors, the dynamic model for 2K-H planetary gear reducer system was established and solutions of the time and frequency domains was obtained by Zong-de Fang[2]. The pure torsion model for compound planetary gear system on the basis of the lumped mass method was firstly established and the natural characteristics of the system was studied by Kahraman A[3]. The adaptive step-varying gill integral
method was used to solve several steady-state forced responses including harmonic and nonharmonic single cycle, subharmonic, quasi-periodicity and chaos of the planetary gear system under varies parameters by Zhimin Sun[4]. Based on torsional-vibratory models, influences of the rotate speed and gear clearance on the bifurcation characteristics of the planetary gear system was studied with the numerical integral method by Tongjie Li[5]. Considering several factors including the friction, time-varying mesh stiffness, gear clearance and comprehensive meshing error, the dynamic model was established and solved by step-varying Gill method and the dynamic responses of friction between gear tooth surfaces was obtained by Enyong Zhu[6]. Considering the effects of gear translation movement on the pressure angle and contact ratio, the lateral-torsional dynamic model to study the dynamic properties of the gear system was established by Woohyung Kim[7]. Nonlinear properties of the planetary gear transmission system such as chaos and bifurcation were studied by and dynamic responses was obtained with the lumped mass and finite element methods by Robert G. Parker[8].

In summary, a series of studies on the planetary gear transmission system dynamics have been conducted by a number of scholars at home and abroad which has formed a relatively complete system of the dynamic theory. However, most of models established by scholars are purely torsional or translational-torsional coupled dynamic models and the uneven stress along the tooth width was ignored. Based on previous studies, considering the time-varying meshing stiffness, time-varying gear clearance, time-varying pressure angels and the swing of gears, the lateral-torsional-swing coupled nonlinear dynamic model was built. In addition, the effects of different supporting stiffness on dynamic responses of the system have been studied under the given working conditions.

THE PLANETARY GEAR SYSTEM AND ITS SOLID MODEL

The research object-2K-H type planetary gear reducer with involute spur gear consists of a sun gear, a ring gear, a carrier and three planetary gears. The geometric parameters of the system are shown in Table I. The three-dimensional model is established according to these geometric parameters in Table I, and its solid model was shown in the Figure 1.
Figure 1. The three-dimensional model of 2K-H planetary gear system.

Table I. Geometric parameters of the planetary gear transmission system.

<table>
<thead>
<tr>
<th>Component</th>
<th>Modulus (mm)</th>
<th>Number of teeth</th>
<th>Tooth width (mm)</th>
<th>Pressure angle (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun gear</td>
<td>0.5</td>
<td>18</td>
<td>12</td>
<td>20</td>
</tr>
<tr>
<td>Planetary gear</td>
<td>0.5</td>
<td>45</td>
<td>12</td>
<td>20</td>
</tr>
<tr>
<td>Ring gear</td>
<td>0.5</td>
<td>108</td>
<td>12</td>
<td>20</td>
</tr>
</tbody>
</table>

THE ESTABLISHMENT OF NONLINEAR DYNAMIC MODEL

The ring gear belongs to the fixed part which has no freedom, while the other components in the system all have five degrees of freedom including rotational, translational and swing freedom. Considering the deformation of bearings and the uneven stress along the tooth width, the simplified dynamic model was shown in Figure 2.

Where \( Oxyz \) was a fixed coordinate with the center of planetary gear transmission system as the origin and at the same time, \( O_{\xi\eta\zeta} \) was a follow-up coordinate with the origin coincided with the axes of the \( i \) th planetary gear and fixed with the carrier. The axes of three follow-up coordinates always stay parallel with the corresponding axes of fixed coordinates.

With the five degrees of freedom of the sun gear and planetary gear represented by symbols \( x_k, y_k, \theta_k, \theta_{k\xi}, \theta_{k\eta}, \theta_{k\zeta} \) \( (k = s, c) \), respectively and the five degrees of freedom of the \( i \) th planetary gear represented by symbols \( \xi_i, \eta_i, \theta_{pi\xi}, \theta_{pi\eta}, \theta_{pi\zeta} \) \( (i = 1, 2, 3) \), the generalized displacement array \( \vec{q} \) for the system can be expressed as

\[
\vec{q} = \{x_s, y_s, \theta_{sx}, \theta_{sy}, \theta_{sz}, x_c, y_c, \theta_{cx}, \theta_{cy}, \theta_{cz}, \xi_1, \eta_1, \theta_{p1\xi}, \theta_{p1\eta}, \theta_{p1\zeta}, \xi_2, \eta_2, \theta_{p2\xi}, \theta_{p2\eta}, \theta_{p2\zeta}, \xi_3, \eta_3, \theta_{p3\xi}, \theta_{p3\eta}, \theta_{p3\zeta}\}^T
\]
The Establishment of Dynamic Equations

The dynamic equations of the system can be derived from the Lagrange equations with the kinetic and potential energy expressed by the generalized coordinate $q$. The dynamic equations of the sun gear, the $i$th planetary gear and carrier are derived in the same way, so taking the carrier as an example shown as equations (1) to (5).

\[
\begin{align*}
\left( m_c + \sum_{i=1}^{3} m_{p_i} \right) \ddot{x}_c & - \sum_{i=1}^{3} m_{p_i} \left[ \left( \dot{x}_c + r_c \right) \dot{\theta}_{c} - \eta_c \dot{\theta}_c^2 + 2 \dot{\theta}_c \dot{x}_c + \ddot{\eta}_c \right] \sin(\theta_c + \psi_c) \\
& + \left[ \eta_c \dot{\theta}_c + (\dot{x}_c + r_c) \dot{\theta}_c^2 + 2 \dot{\eta}_c \dot{\theta}_c - \ddot{\eta}_c \right] \cos(\theta_c + \psi_c) + k_{c} x_c \\
+ \sum_{i=1}^{3} \left( k_{p_i} f \left( \tau_{p_i} - \tau_{p_i} \right) \frac{\partial f \left( \tau_{p_i} - \tau_{p_i} \right)}{\partial x_c} + k_{p_i} f \left( \tau_{p_i} - \tau_{p_i} \right) \frac{\partial f \left( \tau_{p_i} - \tau_{p_i} \right)}{\partial y_c} \right) &= 0
\end{align*}
\]

\[
\begin{align*}
\left( m_c + \sum_{i=1}^{3} m_{p_i} \right) \ddot{y}_c & - \sum_{i=1}^{3} m_{p_i} \left[ \left( \dot{y}_c + r_c \right) \dot{\theta}_{c} - \eta_c \dot{\theta}_c^2 + 2 \dot{\theta}_c \dot{y}_c + \ddot{\eta}_c \right] \sin(\theta_c + \psi_c) \\
& + \left[ \eta_c \dot{\theta}_c + (\dot{y}_c + r_c) \dot{\theta}_c^2 + 2 \dot{\eta}_c \dot{\theta}_c - \ddot{\eta}_c \right] \cos(\theta_c + \psi_c) + k_{c} y_c \\
+ \sum_{i=1}^{3} \left( k_{p_i} f \left( \tau_{p_i} - \tau_{p_i} \right) \frac{\partial f \left( \tau_{p_i} - \tau_{p_i} \right)}{\partial x_c} + k_{p_i} f \left( \tau_{p_i} - \tau_{p_i} \right) \frac{\partial f \left( \tau_{p_i} - \tau_{p_i} \right)}{\partial y_c} \right) &= 0
\end{align*}
\]

\[
\begin{align*}
L_c \ddot{\theta}_c + L_c \omega_c \dot{\theta}_c + \sum_{i=1}^{3} k_{d_{p_i} \psi} \theta_{p_i} \psi_c &= 0
\end{align*}
\]

\[
\begin{align*}
L_c \ddot{\theta}_c - L_c \omega_c \dot{\theta}_c + \sum_{i=1}^{3} k_{d_{p_i} \psi} \theta_{p_i} \psi_c &= 0
\end{align*}
\]
\[
\begin{aligned}
&\left\{ I_x + \sum_{i=1}^{n} I_{x,i} + \sum_{i=1}^{n} m_c \left[ \eta^2 + (\xi + r_i)^2 \right] \right\} \ddot{\theta}_x - \sum_{i=1}^{n} m_c \left[ [\xi \eta^2 - (\xi + r_i) \eta - 2(x_i \xi + r_i \xi + \eta \eta)] \right] \\
&+ \left[ (\xi + r_i) \dot{x}_i + \eta \dot{y}_i \right] \sin(\theta_{x,i} + \psi_i) + [\eta \dot{x}_i - (\xi + r_i) \dot{y}_i \right] \cos(\theta_{x,i} + \psi_i) + \sum_{i=1}^{n} I_{x,i} \dot{\theta}_{x,i}
\end{aligned}
\]

\[
+ \sum_{i=1}^{n} \left( k_{sp,i} \frac{rf(\tau_{z,i} - \tau_{z,sp})}{\partial \theta_{z,i}} + k_{sp,i} \frac{rf(\tau_{z,i} - \tau_{z,sp})}{\partial \theta_{z,i}} \right) = 0
\]

Where \( m_i \) (\( i = c, p_i \)) stands for the mass of carrier and \( i \)th planetary gear, respectively; \( I_{x,j} \) (\( j = x, y, z \)) are the rotational inertia of carrier, \( k_{wu}, k_{wy}, k_{wu}, k_{wy} \) and \( k_{sp,i}, k_{sp,i}, k_{sp,i}, k_{sp,i} \) represent brace stiffness of the component direction, \( k_{sp,i} \) and \( k_{sp,i} \) representing the meshing stiffness between the sun, ring and the \( i \)th planetary gear ,respectively, = \( \alpha_{sp,i} \) (\( i = s, r \)) represents the displacement along the mesh direction between the sun gear, ring gear and the \( i \)th planetary gear, respectively, \( \alpha_{sp,i} \) (\( i = s, r \)) stands for the time-varying pressure angle between the sun gear, ring gear and the \( i \)th planetary gear, respectively and \( \delta_{sp,i}, \delta_{sp,i} \) are the time-varying gear clearance between the \( i \)th planetary gear and the sun gear , ring gear respectively.

The Determination of Nonlinear Parameters

As shown in Figure 3(a), the time-varying pressure angle and position angle derived from the geometric relationship between the sun and the ith planetary gear can be expressed as equations (6) to (7).

\[
\alpha_{sp,i} = \arccos \frac{r_{bs} + r_{sp,i}}{\sqrt{(x_{sp,i} - x_i)^2 + (y_{sp,i} - y_i)^2}} \quad (i = 1,2,3)
\]

\[
\varphi_{sp,i} = \arctan \frac{y_{sp,i} - y_i}{x_{sp,i} - x_i} \quad (i = 1,2,3)
\]

As well, the time-varying pressure angle and position angle between the sun gear and the \( i \)th planetary gear are depicted in Figure 3(b).
\[ \alpha_{p_i} = \arccos \frac{r_{p_i} - r_{mp}}{\sqrt{x_{p_i}^2 + y_{p_i}^2}} \quad (i = 1, 2, 3) \]  

\[ \varphi_{p_i} = \arctan \frac{y_{p_i}}{x_{p_i}} \quad (i = 1, 2, 3) \]

Displacement along the line of action between the sun gear and the \( i \)th planetary gear can be derived from the equations of the time-varying pressure angle and the position angle, shown as

\[ \tilde{s}_{sp_i} = r_{s_i} (\theta_{sz} - \theta_{cz}) + r_{mp} \theta_{p_{cx}} + \left[ x_{p_i} - x_{p_i} + r_{s_z} \cos(\theta_{cz} + \psi_i) \right] \sin(\alpha_{q_{pi}} - \varphi_{q_{pi}}) \]

\[ + \left[ y_{p_i} - y_{p_i} + r_{s_z} \cos(\theta_{cz} + \psi_i) \right] \cos(\alpha_{q_{pi}} - \varphi_{q_{pi}}) \]  

(10)

Similarly, the displacement along the line of action between the ring gear and the \( i \)th planetary gear can also be given as

\[ \tilde{s}_{rp_i} = -r_{s_i} \theta_{cz} - r_{mp} \theta_{p_{cx}} + \left[ x_{p_i} - r_{s_z} \cos(\theta_{cz} + \psi_i) \right] \sin(\alpha_{q_{pi}} + \varphi_{q_{pi}}) \]

\[ - \left[ y_{p_i} - r_{s_z} \sin(\theta_{cz} + \psi_i) \right] \cos(\alpha_{q_{pi}} + \varphi_{q_{pi}}) \]  

(11)

\[ f \left( \tilde{s}_{pq}, \tilde{b}_{pq} \right) (t = s, r) \] is the nonlinear function related to the clearance, shown as

\[ f \left( \tilde{s}_{pq}, \tilde{b}_{pq} \right) = \begin{cases} \tilde{s}_{pq} - \tilde{b}_{pq} & \tilde{s}_{pq} > \tilde{b}_{pq} \\ 0 & \tilde{s}_{pq} < \tilde{b}_{pq} \end{cases} \]

(12)

\[ f \left( \tilde{s}_{pq}, -\tilde{b}_{pq} \right) = \begin{cases} -\tilde{s}_{pq} + \tilde{b}_{pq} & \tilde{s}_{pq} < -\tilde{b}_{pq} \end{cases} \quad (t = s, r) \]

SOLUTION AND RESULT ANALYSIS OF THE DYNAMIC EQUATIONS

The dynamic equations for the system are solved by the fifth-order step-varying adaptive Runge-Kutta numerical integration method which is usually applied to solve the first order differential equations. However, the dynamic equations are the second-order equations, so it’s necessary to transform them into the first order differential equations via state variables \( \tilde{X} \) expressed as

\[ \tilde{X} = \left\{ x_1, x_2, x_3, x_4, ..., x_{47}, x_{48}, x_{49}, x_{50} \right\}^T \]

\[ = \left\{ x_s, \dot{x}_s, y_s, \dot{y}_s, \theta_{sz}, \dot{\theta}_{sz}, \theta_{sy}, \dot{\theta}_{sy}, \theta_{sx}, \dot{\theta}_{sx}, \theta_{phi}, \dot{\theta}_{phi}, \theta_{psi}, \dot{\theta}_{psi}, \alpha_{p_{xi}}, \varphi_{p_{xi}}, \alpha_{q_{xi}}, \varphi_{q_{xi}}, \alpha_{p_{yi}}, \varphi_{p_{yi}}, \alpha_{q_{yi}}, \varphi_{q_{yi}}, \alpha_{p_{zi}}, \varphi_{p_{zi}}, \alpha_{q_{zi}}, \varphi_{q_{zi}}, \alpha_{p_{si}}, \varphi_{p_{si}}, \alpha_{q_{si}}, \varphi_{q_{si}}, \alpha_{p_{di}}, \varphi_{p_{di}}, \alpha_{q_{di}}, \varphi_{q_{di}} \right\}^T \]

Expressions for the second derivative of all the variables are derived via simultaneous equations and reduced the order with state variables introduced.

During the process of the simulation, key parameters in the planetary gear system are given, such as base radius of the sun, ring and planetary gear can be expressed as

\[ r_{bs} = 8.46 \text{mm}, \quad r_{br} = 50.74 \text{mm}, \quad r_{bp} = 21.14 \text{mm}, \]

respectively, the radius of the carrier is \( r_c = 31.5 \text{mm} \), the mass of all of the components can be expressed as \( m_s = 0.22 \text{kg}, \)

\( m_r = 0.36 \text{kg}, \quad m_{pi} = 0.40 \text{kg}, \) the initial gear clearance can be given as \( b_{spi} = b_{mpi} = 25 \mu \text{m}, \)
and all of the moment of inertias can be indicated as $I_{xx} = 8.93 \times 10^{-3}$, $I_{yy} = 5.14 \times 10^{-4}$, $I_{zz} = 2.77 \times 10^{-4}$, $I_{x} = I_{y} = 2.35 \times 10^{-4}$, $I_{x} = 2.27 \times 10^{-3}$ and $I_{y} = I_{z} = 1.56 \times 10^{-5}$ kg m$^2$.

The fifth-order step-varying adaptive Runge-Kutta numerical integration method is adopted to solve the dynamic equations and obtain the steady responses for the system with the moment $M = 40$ Nm applied on the input shaft of the sun gear.

The lateral degrees of freedom is induced by the deformation of bearings, so that the values of dynamic responses with different supporting stiffness can be obtained to study the effects of the supporting stiffness on dynamic responses.

Only the vibration along the $y$ direction of the sun gear and carrier and the vibration along the $\eta$ direction of the first planetary as well as its corresponding phase plane portrait are shown when the value of supporting stiffness for the sun gear, planetary gear and carrier is $2.2 \times 10^6$ N/m. The steady responses for the system are given from Figure 4 to Figure 6.

As shown from Figure 4 to Figure 6, compared with the sun gear and carrier, the amplitude of the planetary is much larger and vibrates more seriously. The amplitude of the planetary decreased slightly with the supporting stiffness added up to $2.2 \times 10^6$ N/m, which is helpful to improve the properties of even loads. In practice, the value of the stiffness for the planetary is not allowed too small, for the power is passed to the carrier via the planetary. As far as the sun gear and the carrier, the increase of the supporting stiffness has little effect on the amplitude along $y$ direction, while the periodicity of dynamic responses for the system is more obvious.

The phase plane portraits of the sun and planetary gear are expressed as closed and roughly elliptical curve bands with some width, which demonstrates the responses are quasi-periodic, when the supporting stiffness is $2.2 \times 10^6$ N/m. Compared with the sun and planetary, phase plane portrait of the carrier is less regular, but the response is still quasi-periodic with the loop curve and properties of the time-domain figure.

(a) Displacement-time responses of sun gear  (b) Velocity-displacement phase plane of sun gear

Figure 4. Simulation results of sun in $y$ direction.
Figure 5. Simulation results of carrier in y direction.

(a) Displacement-time responses of carrier
(b) Velocity-displacement phase plane of carrier

Figure 6. Simulation results of the first planetary gear in \( \eta \) direction.

(a) Displacement-time responses of the first planetary gear
(b) Velocity-displacement phase plane of the first planetary gear

However, with the supporting stiffness increasing to \( 2.2 \times 10^6 \) N/m, the phase plane portraits of the planetary gear and carrier become more regular as quasi-periodic responses, while quasi-periodic response of the sun gear changes into the single-cycle simple harmonic vibration response.

The results above have shown a trend of the dynamic responses changing from the quasi-periodicity to the periodicity for the system with the increased stiffness properly. As far as the feed system of the machine tool, although the quasi-periodic vibration can be defined, but it’s not likely easy to find the regular of the quasi-periodic vibration like the periodic vibration, which means it’s more helpful to eliminate the vibration with the quasi-periodic vibration converted into the periodic vibration in order to improve the machining precision of the machine tool.

CONCLUSIONS

In this study, considering the nonlinear factors including the time-varying meshing stiffness, time-varying gear clearance and time-varying pressure angles, etc, the lateral-torsional-swing coupling nonlinear dynamic model for 2K-H type planetary gear reducer is established by the lumped mass method. Also, the effects of support stiffness on different dynamic responses with Runge-Kutta numerical integration method solved is explored, which has laid a foundation for further researching the dynamic properties in the future.
(1) Based on the previous models, the model built with swing freedom induced by the uneven stress along the tooth width, time-varying pressure angles and time-varying gear clearance induced by lateral movement considered, is more closely to practical operating conditions of transmission system, which are meaningful to further analyze dynamic characteristics of planetary gear transmission system in the future.

(2) Dynamic equations for the system are solved by the fifth-order step-varying adaptive Runge-Kutta numerical integration method and dynamic responses of different supporting stiffness for the system are compared with each other both have a guiding significance to choose the proper bearing.

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REFERENCES