Notes on Logarithmic Function and Complex Powers

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Abstract. Theory of complex function is an important branch of mathematics. It has important application in a lot of natural sciences. Basic research content in the complex function is logarithmic function. This paper studies the definition of logarithmic function, and starting from the definition, it is obtained that logarithmic of complex product equal the sum of logarithmic and other basic properties. As accessory of logarithmic function, we guide the strict definition of the complex power.

Introduction

We would like to define the logarithm[1, 2] in such a way that our definition will agree with the usual definition of log\(x\) when \(x\) is located on the positive part of the real axis[3, 4]. In the real case we can define the logarithm as the inverse of the exponential (that is, log\(x=\gamma\) is the solution of \(e^\gamma=x\)). When we allow \(z\) to range over complex set \(C\) we must be more careful because, as we know, the exponential[5] is periodic and thus does not have an inverse. Furthermore, the exponential is never zero and so we cannot expect to be able to define the logarithm at zero. Thus we must be more careful in our choice of the domain in \(C\) on which we can define the logarithm. Theorem 1 indicates how this may be done.

Definition and Its Feasibility

Theorem 1. Let \(A_{y_0}\) denote the set of complex numbers \(x+iy\) such that \(y_0 \leq y < y_0 + 2\pi\); symbolically,

\[ A_{y_0} = \{ x+iy \mid x \in \mathbb{R} \text{ and } y_0 \leq y < y_0 + 2\pi \}. \]

Then \(e^z\) maps \(A_{y_0}\) in a one-to-one manner onto the set \(\{0\}\).

Note. Notice that a map is one-to-one when the map takes every two distinct points to two distinct points; in other words, two distinct points never get mapped to the same point. The statement that a map is onto a set \(B\) means that every point of \(B\) is the image of some point under the mapping. The notation \(\{0\}\) means the whole plane \(C\) minus the point 0; that is, the plane with the origin removed.

Proof. If \(e^{zi} = e^{z_2}\), then \(e^{zi-z_2} = 1\), and so \(z_1 - z_2 = 2\pi in\) for some integer \(n\). But because \(z_1\) and \(z_2\) both lie in \(A_{y_0}\), where the difference between the imaginary parts of any points is less than \(2\pi\), we must have \(z_1 = z_2\). This argument shows that \(e^z\) is one-to-one. Let \(w \in C\) with \(w \neq 0\). We claim the equation \(e^z = w\) has a solution \(z\) in \(A_{y_0}\). The equation \(e^{zi} = w\) is then equivalent to the two equations \(e^z = |w|\) and \(e^{iy} = w/|w|\). The solution of the first equation is \(x = \log|w|\) where “log” is the ordinary logarithm defined on the positive part of the real axis. The second equation has infinitely many solutions \(y\), each differing by integral multiples of \(2\pi\), but exactly one of these is in the interval \([y_0, y_0 + 2\pi]\). This \(y\) is merely \(\arg w\) where the specified range for the arg function is \([y_0, y_0 + 2\pi]\). Thus \(e^z\) is onto \(\{0\}\).
The sets denoted in Theorem 1 may be depicted as in Figure 1. Here $e^z$ maps the horizontal strip between $y_0$ and $y_0 + 2\pi i$ one-to-one onto $C \setminus \{0\}$.

Note. The notation $z \to f(z)$ is used to indicate that $z$ is sent to $f(z)$ under the mapping $f$.

In Theorem 1 an explicit expression was also derived for the inverse of $e^z$ restricted to the strip $y_0 \leq \text{Im} z < y_0 + 2\pi$ and this expression is stated formally in the following definition.

Definition 1. The function $\log: \{0\} C \to C$, with range $\{0\} \subset C$, is defined by

$$\log z = \log |z| + i \arg z,$$

where $\arg z$ takes values in $[y_0, y_0 + 2\pi)$. Here $\log |z|$ is the usual logarithm of the positive real number $|z|$.

This function is sometimes referred to as the “branch of the logarithm function lying in $\{x + iy | y_0 \leq y < y_0 + 2\pi\}$.”

But we must remember that the function $\log z$ is only well defined when we specify an interval of length $2\pi$ in which $\arg z$ takes its values, that is, when a Figure 1 $e^z$ as a one-to-one function onto $C \setminus \{0\}$. Specific branch is chosen. For example, suppose that the specified interval is $[0, 2\pi)$. Then

$$\log (1 + i) = \log \sqrt{2} + i \frac{\pi}{4}.$$  

However, if the specified interval is $[\pi, 3\pi)$, then $\log (1 + i) = \log \sqrt{2} + i \frac{9\pi}{4}$.

Construction of Logarithm

Theorem 2 summarizes our construction of $\log z$.

Theorem 2. $\log z$ is the inverse of $e^z$ in the following sense: For any branch of $\log z$, we have $e^{\log z} = z$, and if we choose the branch lying in $y_0 \leq y < y_0 + 2\pi$,

then

$$\log(e^z) = z \text{ for } z = x + iy$$

and

$$y_0 \leq y < y_0 + 2\pi.$$  

Note. Usually the domain of $\log$ will be restricted further to

$y_0 \leq y < y_0 + 2\pi$

so that $\log z$ will be a continuous function.

Proof. Since $\log z = \log |z| + i \arg z$,

$$e^{\log z} = e^{\log |z| + i \arg z} = |z| e^{i \arg z} = z$$

Conversely, suppose that $z = x + iy$ and
\( y_0 \leq y < y_0 + 2\pi \).

By definition, 
\[ \log e^z = \log |e^z| + i \arg e^z, \]
But \( |e^z| = e^x \) and \( \arg e^z = y \) by our choice of branch. Thus 
\[ \log e^z = \log e^x + i y = x + iy = z. \]

The logarithm defined on \( C \setminus \{0\} \) behaves the same way with respect to products as the logarithm restricted to the positive part of the real axis. This is proved in Theorem 3.

**Theorem 3.** If \( z_1, z_2 \in C \setminus \{0\} \), then 
\[ \log(z_1 z_2) = \log z_1 + \log z_2 \]
(up to the addition of integral multiples of \( 2\pi i \)).

\[ \arg(z_1 z_2) = \arg z_1 + \arg z_2 \]

**Proof.** \( \log(z_1 z_2) = \log |z_1 z_2| + i \arg(z_1 z_2) \) where an interval \( [y_0, y_0 + 2\pi) \) has been chosen for the values of the \( \arg \) function. We know that 
\[ \log(z_1 z_2) = \log |z_1| |z_2| + i \arg z_1 + \log |z_2| \]
and 
\[ \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \]
(up to integral multiples of \( 2\pi i \)). Thus 
\[ \log(z_1 z_2) = \log z_1 + i \arg z_1 + \log z_2 \]
(up to integral multiples of \( 2\pi i \)).

To illustrate: Let us find where the range for the \( \arg \) function is chosen as, say, \([0, 2\pi)\). Thus 
\[ \log((-1 - i)(1 - i)) = \log(2) = 2 + \pi i. \]

On the other hand, 
\[ \log(-1 - i) = \log \sqrt{2} + i \frac{5\pi}{4} \]
and 
\[ \log(1 - i) = \log \sqrt{2} + i \frac{7\pi}{4} \]
Thus 
\[ \log(-1 - i) + \log(1 - i) = \log 2 + i 3\pi = (\log 2 + \pi i) + 2\pi i; \]
so in this case, when \( z_1 = -1 - i \) and \( z_2 = 1 - i \), \( \log(z_1 z_2) \) differs from \( \log z_1 + \log z_2 \) by \( 2\pi i \).

**Complex Powers**

We are now in a position to define the term \( a^b \) where \( a, b \in C \) and \( a \neq 0 \) (read “\( a \) raised to the power \( b \)”). Of course, however we define \( a^b \), the definition should reduce to the usual one in which \( a \) and \( b \) are real numbers. The trick is to notice that \( a \) can also be written \( e^{\log a} \) by Theorem 2. If \( b \) is an integer, we have \( a^b = (e^{\log a})^b = e^{b \log a} \). This last equality holds since if \( n \) is an integer and \( z \) is any complex number, by Theorem 8(i). Thus we are led to formulate the following definition.

**Definition 2.** \( a^b \) (where \( a, b \in C \) and \( a \neq 0 \)) is defined to be \( e^{b \log a} \); it is understood that some interval \( [y_0, y_0 + 2\pi) \) can be chosen. (That is, some branch of \( \log \) has been chosen within which the \( \arg \) function takes its values.

**Conclusions**

It is of the utmost importance to understand precisely what this definition involves. Note especially that in general \( \log z \) is “multiple-valued”; that is, \( \log z \) can be assigned many different values because different intervals \([y_0, y_0 + 2\pi)\) can be chosen.
References

[1] Mingyang Lv, Xiaogang Zhang, Hua Chen. Comparison of methods based on phase space reconstruction for prediction of Sintering Temperature time series in Rotary Kiln [A]. The 28th China process control conference 2017 (CPCC) and commemorate the 30th anniversary of China process control meeting the set [C]. 2017, 56-61


