Mean-variance Portfolio Selection Problem with Vasicek Stochastic Interest Rates

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Keywords: mean-variance portfolio selection; Vasicek stochastic interest rate; efficient frontier; stochastic control.

Abstract: This paper studies a continuous-time mean-variance portfolio selection problem with Vasicek stochastic interest rates. Compared with the mean-variance model with deterministic interest rate, a verification theorem without the classical Lipschitz and growth conditions is required to solve our portfolio selection problem. By using the stochastic dynamic programming principle and Hamilton-Jacobi-Bellman equation approach, the optimal investment strategy, the value function and the efficient frontier are derived in closed-form.

1. Introduction

Since the pioneering work \cite{1} in the single-period mean-variance formulation emerged, the mean-variance portfolio selection problem has provided a fundamental basis for portfolio construction and has stimulated hundreds of extensions and applications. \cite{2} and \cite{3} are the first to extend the original static mean-variance model to multi-period and continuous-time cases respectively. Since then, various kinds of problems under mean-variance criterion have been investigated analytically. For instances, \cite{4} and \cite{5} considered an asset and liability management problem. \cite{6} and \cite{7} solved a portfolio selection problem when stock prices follow jump diffusion processes. \cite{8} and \cite{9} studied mean-variance optimization problems for an insurer. \cite{10} and \cite{11} investigated mean-variance portfolio optimization problems with no-shorting constraint under continuous-time and discrete-time settings, respectively.

Most literatures concentrate on the continuous-time cases under the assumption that the interest rate is deterministic. But the interest rate in the real world is affected greatly by some uncertain affairs, such as inflation, war and disaster. Therefore, it is more realistic to assume that the interest rate is a stochastic process rather than a deterministic one. Stochastic interest rate models are widely applied in option pricing, hedging and portfolio optimization. In the past years, many scholars have devoted themselves to the optimal portfolio selection problem with stochastic interest rate. For examples, \cite{12} considered an investment problem to maximize the power-utility from terminal wealth under the assumption that the interest rate is a stochastic process. \cite{13} investigated an optimal investment-consumption problem with stochastic interest rate, and they considered one example where the interest rate is modeled as Vasicek model. \cite{14} studied a power-utility maximization problem with stochastic interest rate and stochastic volatility described by Cox-Ingersoll-Ross (CIR) model. \cite{15} obtained an optimal consumption and investment policy by an asymptotic method under the assumption that both the volatility and interest rate vary according to ergodic Markov diffusion processes. \cite{16} solved the continuous-time mean-variance investment problem with stochastic uniformly bounded market coefficients. \cite{17} considered an asset and liability management problem in a continuous-time mean-variance framework. Unfortunately, the assumption of uniform bounded
coefficients does not include the case of Vasicek stochastic interest model. [18] and [19] considered the portfolio selection problems with regime switching, where the coefficients, including the interest rate, depend on the market states and hence have some randomness since the market states in the future are uncertain. So portfolio optimization problems with regime switching can be classified as problems with stochastic interest rate. All in all, mean-variance portfolio selection models with stochastic interest rates are limited. We will work on this topic and considers a mean-variance portfolio selection problem with Vasicek stochastic interest rate.

The rest of this paper is organized as follows. In Section 2, we formulate the mean-variance portfolio selection model with Vasicek stochastic interest rates and present a new definition of admissible strategies and a new sufficient condition that guarantee a candidate for the value function to be indeed optimal. In Section 3, we obtain the candidates for the optimal strategy and the value function in Vasicek model. In Section 4, we analyze in detail the properties of these candidates. We obtain the efficient frontier in closed-form in Section 5. Section 6 concludes this paper.

2. Problem Formulation

Let $\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0}$ be a fixed filtered complete probability space on which is defined a standard $\mathcal{F}_t$-adapted Brownian motion $W(t)$, here $\mathcal{F}_t = \sigma(W(s): 0 \leq s \leq t)$. An investor joins a market at time 0 with initial wealth $x_0 > 0$ and plans to invest dynamically over a fixed time horizon $T$. For the sake of simplicity, we assume that there are two assets traded in a frictionless market. One asset is a saving account whose price process is

$$
\frac{dB(t)}{B(t)} = B(t) \frac{dr(t)}{r(t)} - \theta(t) dt + b dW(t), \quad B(0) = b_0,
$$

here $d$ represents the differential operator, and the short rate $r(t)$ satisfies the differential equation

$$
\frac{dr(t)}{r(t)} = (\alpha(t) + b \xi(t)) dt + b dW(t), \quad r(0) = r_0,
$$

where $b > 0$ is a constant, the risk premium $\xi(t)$ is a deterministic and continuous function, $W(t)$ is a standard Brownian motion, and $\alpha(t) = \theta(t) - \sigma(t)$ is a stochastic process related to $r(t)$, here $\sigma(t)$ is a deterministic and continuously differentiable function. The interested readers can refer to [20] for detailed information. The other asset is a (zero) bond whose price process is modeled as

$$
\frac{dS(t)}{S(t)} = \left(\frac{r(t) + \xi(t) \sigma(t)}{S(t)}\right) dt + \sigma(t) dW(t), \quad S(0) = s_0,
$$

where $\sigma(t)$ is a deterministic and continuously differentiable function. Here the appreciation rate is $r(t) + \xi(t) \sigma(t) := \mu(t)$, which results from the fact that the risk premium $\xi(t)$ is defined as $\frac{\mu(t) - r(t)}{\sigma(t)}$.

The interested reader can refer to [12] (p.3) for more information.

Let $\pi(t)$ denote the amount invested in the (zero) bond at time $t$ ($t \in [0, T]$), and $X^\pi(t)$, the wealth at time $t$ corresponding to investment strategy $\pi$. Then the wealth process satisfies the following stochastic differential equation:

$$
\frac{dX^\pi(t)}{X^\pi(t)} = \left[\pi(t) \xi(t) \sigma(t) + X^\pi(t) r(t)\right] dt + \pi(t) \sigma(t) dW(t),
$$

with initial condition $X(0) = x_0$.

In this paper, we will discuss the following mean-variance portfolio selection problem

$$
P(w) \min_{\pi \in \Pi(0, x_0)} \mathbb{E}[X^\pi(T) - w]^2,
$$

where $w$ is a pregiven return level, and $\Pi(0, x_0)$ denotes the set of all admissible controls defined as below.
Definition 1 A control \( \pi(t) \) is said to be admissible if

(i) for any initial wealth \( x_0 > 0 \), the stochastic differential equation (SDE for short hereafter) (1) has a unique solution \( X^\pi(t) \) corresponding to \( \pi(t) \);

(ii) for all \( k \in \mathbb{N} := \{1, 2, \cdots \} \), the corresponding solution \( X^\pi(t) \) satisfies \( \mathbb{E}(\sup_{t \in [0,T]} |X^\pi(t)|^k) < +\infty \);

(iii) for all \( k \in \mathbb{N} \), \( \mathbb{E}\left(\int_0^T |\pi(t)|^k \, dt\right) < +\infty \).

By convex optimization theory, problem \( P(w) \) can be solved via the following optimal stochastic control problem with a Lagrange multiplier \( \lambda^* \):

\[
\text{PL}(\lambda, w) = \min_{\pi(t) \in \mathbb{B}(0, \lambda)} \left\{ \mathbb{E}[X^\pi(T) - w]^2 - 2\lambda[\mathbb{E}[X^\pi(T)] - w] \right\}.
\]

The relationship between the optimal solutions of these two problems is concluded in the following lemma (see [21]).

**Lemma 1** Denote by \( \hat{\lambda}(x, t, x(t)) \) and \( \hat{\lambda}(x, t, x(t)), t \in [0, T] \), respectively, the optimal value and the optimal strategy of problem \( \text{PL}(\lambda, w) \). Then the optimal value and the optimal strategy of problem \( P(w) \) are given by \( \lambda^* \) and \( \hat{\lambda}(x, t, x(t)) \), respectively, where \( \lambda^* = \arg\sup_{\lambda \in \mathbb{R}} \Gamma(\lambda) \).

Since the objective function of problem \( \text{PL}(\lambda, w) \) can be rewritten as \( \mathbb{E}[X^\pi(T) - (\lambda + w)]^2 - \lambda^2 \), the solution of problem \( \text{PL}(\lambda, w) \) is equivalent to that of the problem

\[
\text{PL2}(\lambda, w) = \min_{\pi(t) \in \mathbb{B}(0, \lambda)} \mathbb{E}[X^\pi(T) - (\lambda + w)]^2.
\]

For problem \( \text{PL2}(\lambda, w) \), define the value function

\[
V(t, x, r) = \min_{\pi(t) \in \mathbb{B}(0, \lambda)} \mathbb{E}[X^\pi(T) - (\lambda + w)]^2 \mid X(t) = x, r(t) = r.
\]

By using Bellman’s optimality principle for dynamic programming, we can derive

\[
V(t, x, r) = \min_{\pi(t) \in \mathbb{B}(0, \lambda)} \mathbb{E}[V(t+h, X^\pi(t+h), r(t+h))], \quad (\forall h > 0).
\]

Define an operator

\[
A^{v, s}V(t, x, r) = V_x + V_{\pi}(\xi(t)\sigma(t) + x(t)r(t)) + \frac{1}{2} V_{\pi} \sigma^2 + V_{\pi} b \pi(t) \sigma(t) + \frac{1}{2} V_x b^2 + V_x (\alpha(t) + b \xi(t)).
\]

By using the Itô’s formula, the following equation (3) is obtained,

\[
V(t+h, X^\pi(t+h), r(t+h)) = V(t, x, r) + \int_t^{t+h} A^{v, s}V(s, X(s), r(s))ds + \int_t^{t+h} (b V_x + V_{\pi}(\xi(s)) \sigma(s)) dW(s).
\]

According to [22](p.139), if \( E_r \int_t^{t+h} (b V_x + V_{\pi}(\xi(s)) \sigma(s))^2 ds < +\infty \), then \( \int_t^{t+h} (b V_x + V_{\pi}(\xi(s)) \sigma(s)) dW(s) \) is a martingale. Therefore, when \( E_r \int_t^{t+h} (b V_x + V_{\pi}(\xi(s)) \sigma(s))^2 ds < +\infty \), substituting (3) into (2) yields the Hamilton-Jacob-Bellman (HJB for short) equation of \( V(t, x, r) \) as follows:

\[
V_x + V_{\pi}(\xi(t)\sigma(t) + x(t)r(t)) + \frac{1}{2} V_{\pi} \sigma^2 + V_{\pi} b \pi(t) \sigma(t) + V_x b^2 + V_x (\alpha(t) + b \xi(t)) = 0,
\]

with the boundary condition \( V(T, x, r) = (x - (\lambda + w))^2 \).

In order to know the contribution of HJB equation (4) to derive the optimal strategy and the value function, so we derive the following verification theorem.

**Theorem 1** suppose that \( v(t, x, r) \in C^{2,2}([0, T] \times O) \), where \( O \subseteq \mathbb{R}^2 \), satisfies

(i) \( v(t, x, r) \) solves (4) with boundary condition;

(ii) for any admissible control \( \pi(t) \) and its corresponding wealth process,

\[
E_r \int_t^{t+h} (b V_x + V_{\pi}(\xi(s)) \sigma(s))^2 ds < +\infty, \quad \forall t \in [0, T], h > 0;
\]

\[
E_r \int_t^{t+h} (b V_x + V_{\pi}(\xi(s)) \sigma(s))^2 ds < +\infty, \quad \forall t \in [0, T], h > 0;
\]
(iii) for all sequences of stopping times \(\{\tau_n : 0 \leq \tau_n \leq T\}_{n \in \mathbb{N}}\) and any admissible strategy \(\pi(\cdot) \in \Pi(0, \infty)\), the sequence \(\{v(\tau_n, X^\pi(\tau_n), r(\tau_n))\}_{n \in \mathbb{N}}\) is uniformly integrable.

Then we have

(a) \(v(t, x, r) \leq V(t, x, r)\);

(b) if there exists an admissible strategy \(\pi(t) \in \arg\min_r A^\pi(t, X^\pi(t), r)\), then \(v(t, x, r) = V(t, x, r)\).

Next we will work on deducing the optimal strategy and the value function of problem \(PL_2(\lambda, w)\) by Theorem 1.

3. Candidates For The Optimal Solutions

Assume that \(v\) is a solution of HJB equation (4). When \(v_w > 0\), the candidate for the optimal strategy of problem \(PL_2(\lambda, w)\) is

\[
\tilde{\pi}(t) = -\frac{v_r + v \xi(t)}{v_w}. \tag{5}
\]

Inserting equation (5) back into (4) yields

\[
v_r + v_{xx} + \frac{1}{2}v_{rr} + v_r (\hat{a}(t) + b \xi(t)) - \frac{1}{2} \frac{(v_r + v \xi(t))^2}{v_w} = 0. \tag{6}
\]

with terminal condition \(v(T, x, r) = (x - (\lambda + \omega))^2\).

We can verify that \(v\) has the form

\[
v(t, x, r) = P(t, r)x^2 - 2(\lambda + \omega)Q(t, r)x + (\lambda + \omega)^2R(t, r)
\]

with \(P(t, r) > 0\), \(P(T, r) = 1\), \(Q(T, r) = 1\) and \(R(T, r) = 1\). Substituting the above expression into equation (6), we obtain the following partial differential equations for \(P(t, r)\), \(Q(t, r)\) and \(R(t, r)\) respectively:

\[
\begin{cases}
P_r + (\hat{a} + b \xi(t))P_r + 2rP - \frac{(\xi(t)P + bP_r(t, r))^2}{P} + \frac{1}{2} b^2 P_r = 0, \\
P(t, r) > 0, P(T, r) = 1;
\end{cases} \tag{7}
\]

\[
\begin{cases}
Q_r + rQ + (\hat{a} + b \xi(t))Q_r - \frac{(\xi(t)P + bP_r(t, r))(\xi(t)Q + bQ_r)}{P} + \frac{1}{2} b^2 Q_r = 0, \\
Q(T, r) = 1;
\end{cases} \tag{8}
\]

\[
\begin{cases}
R_r(t, r) + (\hat{a} + b \xi(t))R_r(t, r) + \frac{1}{2} b^2 R_r(t, r) - \frac{(\xi(t)Q + bQ_r)^2}{P} = 0, \\
R(T, r) = 1.
\end{cases} \tag{9}
\]

Let \(P(t, r)\) and \(Q(t, r)\) be of the forms:

\[
P(t, r) = e^{A(t) + B(t)}, \tag{10}
\]

\[
Q(t, r) = e^{C(t) + D(t)}, \tag{11}
\]

with terminal conditions

\[
A(T) = B(T) = C(T) = D(t) = 0. \tag{12}
\]

Now we will deduce the expression of \(A(t)\), \(B(t)\), \(C(t)\), and \(D(t)\) from the above equations. Inserting (10) into (7) gives

\[
(A'(t) - \alpha A(t) + 2)r(t) + B'(t) - \frac{1}{2} b^2 A^2(t) + (\hat{a}(t) - b \xi(t))A(t) - \xi^2(t) = 0.
\]

Substituting (11) into (8) results in
Solving these two equations together with terminal conditions (12) results in

\[
A(t) = \frac{2}{\alpha} (1 - e^{-\alpha(T-t)}) = 2C(t),
\]

(13)

\[
B(t) = \int_0^t \left( -\frac{1}{2} b^2 A^2(s) + \left( \delta(s) - b\xi(s) \right) A(s) - \xi^2(s) \right) ds,
\]

(14)

and

\[
2D(t) = \int_0^t \left[ 2\theta(s)C(s) + b^2 C^2(s) - 2b^2 A(s)C(s) - 2(bA(s) + \xi(s))\xi(s) \right] ds.
\]

(15)

It is easy to find that

\[
2D(t) - B(t) = -\int_0^t (\xi(s) + bC(s))^2 ds.
\]

(16)

By using the expressions of \( P(t, r) \) and \( Q(t, r) \), (9) becomes

\[
R(t, r) + (\alpha + b\xi(t))R(t, r) + \frac{1}{2} b^2 R^2(t, r) = (\xi(t) + bC(t))^2 e^{2(0\rightarrow R(t))} = 0.
\]

(17)

So the solution of the above equation can be of the form

\[
R(t) = e^{2(0\rightarrow R(t))} = e^{-\int_0^t (\xi(s) + bC(s))^2 ds}.
\]

(18)

The candidate for the optimal strategy has the form

\[
\hat{\pi}(t, x) = \frac{\lambda + w}{\sigma(t)} \frac{\xi(t) + bC(t)}{\xi(t) + bC(t)} e^{-C(t) + B(t)} - \frac{\xi(t) + bA(t)}{\sigma(t)} x(t)
\]

(19)

and the respective candidate for the value function is of the form

\[
v(t, x, r) = e^{A(t) - B(t)} x - 2(\lambda + w) e^{C(t) + B(t)} x + (\lambda + w^2) e^{-\int_0^t \xi(s) + bC(s) ds} - \int_0^t \xi(s) + bC(s) ds
\]

(20)

with the expressions of \( A(t), B(t), C(t), \) and \( D(t) \) as (13)-(15). In the next section, we will analyze the properties of the candidates \( \hat{\pi}(t, x) \) and \( v \).

4. Properties Of Candidates \( \hat{\pi}(t, x) \) And \( v \)

In order to analyze the properties of the candidates, we first solve the SDE of \( r(t) \) and introduce some relevant lemmas.

In Vasicek model, SDE of \( r(t) \) is

\[
dr(t) = -\alpha r(t) dt + (\theta(t) + b\xi(t)) dt + bdW(t),
\]

with \( r(0) = \xi_0 \). Its unique solution is

\[
r(t) = \xi_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} (\theta(s) + b\xi(s)) ds + \int_0^t be^{-\alpha(t-s)} dW(s),
\]

(21)

by using the variation-of-constant formula for the SDE (p.313 in [23]).

Lemma 2 \( \mathbb{E}(\sup_{t \in [0,T]} e^{\int_0^t h(s)ds}) < +\infty \), for any deterministic and bounded function \( h(t) \).

Proof: Since

\[
\exp\left[ \int_0^T h(s) dW(s) \right] = \exp\left[ \frac{1}{2} \int_0^T h^2(s) ds \right] \exp\left[ -\frac{1}{2} \int_0^T h^2(s) ds + \int_0^T h(s) dW(s) \right]
\]

\[
\leq \exp\left[ \frac{1}{2} \int_0^T h^2(s) ds \right] \exp\left[ -\frac{1}{2} \int_0^T h^2(s) ds + \int_0^T h(s) dW(s) \right],
\]

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where $\exp\left[\frac{1}{2}\int_0^T h^2(s) ds\right]$ is a constant. Let 

\[ Z(t) = \exp\left[ -\frac{1}{2}\int_0^t h^2(s) ds + \int_0^t h(s) dW(s) \right]. \]

Then the SDE of $Z(t)$ is of the form $dZ(t) = Z(t)h(t) dW(t)$ with $Z(0) = 1$. If $h(t)$ is deterministic and bounded, then $|Z(t)h(t)| \leq 2k^2 |Z(t)|^2$ holds for a suitable constant $K > 0$, namely, condition (R) for SDE in [24] (p.77) is satisfied. Then Corollary 10 in [24] (p.85) guarantees $E(\sup_{t\in[0,T]} Z(t)) < +\infty$, which leads to $E(\sup_{t\in[0,T]} e^{\int_0^t h(s) dW(s)}) < +\infty$.

**Lemma 3** $\int_0^T |r(t)| dt < +\infty$ a.s.

**Proof:** It is easy to show that the SDE of $r(t)$ satisfies the condition (R) in [24] due to the bounded coefficients, which results in $E(\max_{0 \leq t \leq T} |r(t)|^p) < +\infty$ by corollary 10 in [24] for any $\rho \in \mathbb{N}$. Hence

\[ E(\int_0^T |r(t)| dt) \leq T E(\max_{0 \leq t \leq T} |r(t)|) < +\infty, \]

which implies $\int_0^T |r(t)| dt < +\infty$ a.s.

**Lemma 4** For any integer $m$,

\[ E(\sup_{t\in[0,T]} e^{2m\int_0^t r(s) ds}) < +\infty \]

**Proof:** Substituting (21) into $\int_0^T r(s) ds$ yields

\[ 2m \int_0^T r(s) ds = 2m \int_0^T (\xi e^{-\alpha s} + \int_0^s e^{-\alpha(t-s)}(\theta(u) + b\xi(u))du + \int_0^s be^{-\alpha(t-s)} dW(u) ds \]

\[ \equiv \Gamma(t) + 2mb \int_0^T (\int_0^s e^{-\alpha(t-s)} ds) dW(u) \equiv \Gamma(t) + \frac{2mb}{\alpha} \int_0^T (1 - e^{-\alpha(t-s)}) dW(u), \]

where $\Gamma(t)$ is a deterministic and bounded function in $[0,T]$. Hence we only need to consider the term $e^{\int_0^t h(s) dW(s)}$, where $h$ is a deterministic and bounded function in $[0,T]$. According to Lemma 2, the conclusion of the lemma is true.

Now we will discuss the properties of $\hat{\pi}(t,x)$ and $v$. Based on Lemma 1-Lemma 4, we can prove the following theorems: Theorem 2 – Theorem 4.

**Theorem 2** $\hat{\pi}(t,x)$ in (19) is admissible.

**Theorem 3** The candidate $v$ for value function satisfies all the conditions in Theorem 1.

**Theorem 4** For problem $PL2(\lambda, w)$, the optimal strategy is

\[ \hat{\pi}(t,x(t)) = \frac{(\lambda + w)(\xi(t) + hC(t))}{\sigma(t)} e^{-\alpha(t)\int_0^t h(s) ds} - \frac{\xi(t) + hC(t)}{\sigma(t)} x(t), \quad t \in [0,T]. \]

and the value function is

\[ V(t,x,r) = e^{\alpha(t)\int_0^t h(s) ds} x^2 - 2(\lambda + w)e^{\alpha(t)\int_0^t h(s) ds} x + (\lambda + w)^2 e^{\int_0^t (\alpha(t)\int_0^s h(u) du)^2} , \quad t \in [0,T]. \]

where $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are given by (13)-(15) for Vasicek model.

**5. Optimal Solutions And Efficient Frontier Of Problem $P(w)$**

According to the relationship between $PL1(\lambda, w)$ and $PL2(\lambda, w)$, if we define $\Gamma(\lambda)$ as the optimal objective function of $PL1(\lambda, w)$, then
\[ \Gamma(\lambda) = P(0, r_0)x_0 - 2wQ(0, r_0)x_0 + w^2R(0) + \lambda^2(R(0) - 1) + 2\lambda(wR(0) - Q(0, r_0)x_0). \]

Noting that \( R(0) = e^{-\int_{t_0}^{t} (\sigma(t) + \xi(t))dt} \leq 1 \), then \( \lambda^* = \arg\max_{\lambda \in \mathbb{R}} \Gamma(\lambda) \) exists and is
\[
\lambda^* = \frac{wR(0) - Q(0, r_0)x_0}{1 - R(0)}.
\]

Now according to the relationship between \( P(w) \) and \( PL(w) \) in Lemma 1, the optimal strategy and the efficient frontier of problem \( P(w) \) are concluded in theorem 5 below.

**Theorem 5** For problem \( P(w) \), the optimal strategy is
\[
\pi^*(t, x(t)) = \hat{\pi}(t, x(t)) = \frac{x - Q(0, r_0)x_0}{1 - R(0)} \frac{bC(t) + \xi(t)Q(t, r)}{P(t, r)} \frac{bA(t) + \xi(t)x(t)}{\sigma(t)} x(t),
\]
and the efficient frontier is
\[
\text{Var}(X(T)) = \frac{R(0)}{1 - R(0)} \left[ E\left(\frac{Q(0, r_0)x_0}{R(0)} \right) \right]^2,
\]  
where \( R(0) = e^{-\int_{t_0}^{t} (\sigma(t) + \xi(t))dt} \).

6. **Conclusions**

In this paper, a continuous-time mean-variance portfolio model with Vasicek stochastic interest rates has been investigated. The assumption of stochastic interest rates is more realistic but results in difficulties for handling the problem. In contrast to the model with a deterministic interest rate, our wealth process does not satisfy the classical Lipschitz and growth conditions, and the wealth process corresponding to a candidate for the optimal strategy does not follow a homogeneous differential equation any more, which leads to more difficulties in deriving the analytic solution of the wealth process. But we have solved the problem successfully by constructing an auxiliary variable and deriving its analytic expression. Also the ideas and methods in this paper may throw some lights on mean-variance problems with other stochastic interest rates or stochastic volatility rates, such as Heston stochastic volatility, CIR interest rate, etc.

**Acknowledgements**

This research is supported by Guangdong Philosophy and Social Science Planning Project (GD13XGL41) and Science and Technology Planning Project of Guangdong Province, China (2017A040406024).

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