The Pricing of Small and Medium-sized Enterprise Collective Bonds with Embedded Option Based on the Copula Function

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\textbf{Abstract.} Because of collective bonds of small and medium-sized enterprises have several sub-issuers, the method of pricing is different from common corporate bonds to a great extent. In this paper, the Copula function is used to compute the united default distribution of the sub-issuers of small and medium-sized enterprises collective bond, and the option of the bond is being considered in the modeling, so the bond’s credit spread and price can be calculated correctly.

1. Introduction

In recent years, with the continuous development of China’s bond market, a new form of corporate bonds—small and medium enterprise collective bonds (hereinafter referred to as the “collective bonds”) emerged in the bond market. Compared with ordinary corporate bonds, collective bonds have several sub-issuers, and their credit ratings are relatively low. Therefore, the pricing of collective bonds need to be improved on the basis of the traditional bond pricing method in theory. The comprehensive calculation of the default risk of sub-issuers, the impact on the overall credit rating of the collective bonds and the value of the issue of embedded options should be considered.

2. Literature Review

The most basic pricing method of corporate bonds is Merton’s structured model and Jarrow [1], Turnbull’s the simplified model [2], and on the basis of the two models, scholars developed many other pricing methods. Building a credit risk model to determine the bond’s credit spreads is the key to all of these models. Classic credit risk models are Z-Score model of Altman, ZETA model of Altman, and Logit model of Ohlson [3-5].

The above studies mainly focus on the pricing of default risk for a single issue, and the related model can not be directly applied to collective bond pricing in China. The reasons are: (1) the joint probability distribution assumes that all the variables have the same correlation structure, but in reality, the returns of different enterprises don’t have the same correlation structure; (2) the existing joint probability distribution model can combine the marginal distribution and the correlation structure, and reduce the flexibility of the model, and increase the difficulty of parameter estimation. Therefore, it is necessary to find a method to link the default distribution of a single subject and the distribution of the multiple subjects’ joint default.

The Copula function allows the different variables to have different correlation structures, and different edge distributions from the correlation structure, which increases the flexibility of the modeling, and reduces the difficulties of parameter estimation.

In this paper, Copula connection function is used to measure the distribution of different subject’s joint default of the small and medium enterprise collective bonds, so as to determine the credit spread. Furthermore, the option embedded in the collective bonds will be in the full consideration.
3. Construction of Pricing Model

3.1 Calculation of default rate of a single issuer

As the issuers of the collective bonds are non-listed companies, thus, for the calculation of the default rate of a single issuer, we can use factor analysis and Logit model.

3.1.1 Using factor analysis to get factors which influence single issuer’s default rate

In this paper, the factor analysis model is adopted to reduce dimensionality and transform P financial indices which are multicollinearity into other uncorrelated factors $B$. Each factor $B$ is a linear combination of various financial indices, and is not related to each other. Specific expressions are:

$$
\begin{align*}
B_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}x_p \\
B_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2p}x_p \\
&\vdots \\
B_k &= a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kp}x_p
\end{align*}
$$

Where $k$ is less than $p$, we just need to choose a small numbers of $F$ to replace the original financial variable $x$.

3.1.2 Construction of Logit model

Logit model is the two element selection model that assumes the probability of the occurrence of the event meets the standard Logistic cumulative probability function. Specifically, the corporate default probability obeys the following Logit model:

$$
\ln\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1B_1 + \beta_2B_2 + \ldots + \beta_kB_k, \quad p = \frac{\exp\left(\beta_0 + \sum_{i=1}^{k} \beta_iB_i\right)}{1 + \exp\left(\beta_0 + \sum_{i=1}^{k} \beta_iB_i\right)}
$$

Formula (2) represents the condition of next period default probability under the situation that previous financial factors ($B_1, B_2, \ldots, B_k$) of single issuer are known. The factors ($B_1, B_2, \ldots, B_k$) result from the uncorrelated factors obtained from the former part of the factor analysis model.

3.1.3 Calculation of default intensity and default function

The most classical default function is the survival probability function solved by the simple method. Using this function, the default function (death function) of sub-issuers in the collective bond can be expressed as:

$$
F(t) = 1 - e^{-ht}
$$

(3)

Formula (3) represents the default probability of a subject from the beginning of the 0 moment to $t$ moment, where $h$ represents the default intensity.

For the default intensity $h$, we can get it from a single enterprise default $p$ in the former two parts:

$$
h = -\ln(1-p)
$$

(4)

3.2 Using copula function to calculate the joint default rate

3.2.1 Copula connection structure

Copula function has a very important theorem - Sklar theorem: assuming there exists n-dimensional random vector $(\xi_1, \xi_2, \ldots, \xi_n)$, its marginal distribution function and joint distribution function respectively are $F_i(x_i)$ ($i=1,2,\ldots,n$) and $F(x_1, x_2, \ldots, x_n)$, it exists n-dimensional connection structure $C : [0,1]^n \rightarrow [0,1]$, it makes:

$$
F(x_1, x_2, \ldots, x_n) = C\left(F(x_1), F(x_2), \ldots, F(x_n)\right)
$$

(5)
If the marginal distribution and joint distribution functions are differentiable, the joint probability density function of n-dimensional random vector \((\xi_1, \xi_2, \ldots, \xi_n)\) can be expressed as:

\[
f(x_1, x_2, \ldots, x_n) = f_1(x_1) \times f_2(x_2) \times \cdots \times f_n(x_n) \times c(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))
\]

where \(f_i(x_i)\) is the probability density function of \(F_i(x_i)\), and it has:

\[
c(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) = \frac{\partial^n c(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))}{\partial F_1(x_1) \partial F_2(x_2) \cdots \partial F_n(x_n)}
\]  \(\text{(6)}\)

From formula (6), in the case of the probability distribution function of the marginal probability distribution of the different sub-issuers of the collective bond, the probability distribution of the joint default can be obtained by using a connection function. Sklar theorem provides a possible way: to find a Copula function, which connects the marginal probability distribution functions of sub-issuers, so as to obtain the joint default probability of all the issuers.

In order to calculate conveniently, this paper chooses a normal Copula connection structure which contains only one parameter, and the joint default probability of all the issuers can be expressed as:

\[
P(T_1 < t, T_2 < t, \ldots, T_n < t) = C(u_1, u_2, \ldots, u_n) = \frac{1}{\rho} \exp \left[ -\frac{1}{2} \xi^T (\rho - I) \xi \right]
\]  \(\text{(7)}\)

where \(\xi = (\xi_1, \xi_2, \ldots, \xi_n)\), \(\xi_i = \Phi^{-1}(u_i) = \Phi^{-1}(F_i(t)), i=1,2,\ldots,n, \rho \in (-1,1)\), it stands for the correlation parameter matrix, that is the default correlation coefficient matrix among different issuers.

### 3.2.2 Parameter solution of normal Copula

The specific process is as follows:

a. Obtain the financial data of \(m\) time point before the collective bonds issued by the default issuers (\(n\)). For the default issuers who lack of historical financial data, we consider selecting the listing corporations who have the same industry (the main business is similar), the same size, the same region to substitute them.

b. Use the factor analysis and Logit model described in the previous paper to find out the default intensity of each issuer in each period, so as to obtain the matrix \(H(m \times n)\) of the each issuer and their default intensity in each period.

c. Find out the linear correlation coefficient matrix of each column of the matrix \(H\), and use it to substitute \(\rho\) approximately.

### 3.2.3 Simplification and modification of the model

This paper proposes a new method to simplify the calculation. The specific principle is to regard the two default issuers who have the largest correlation coefficient as a whole, and use two normal Copula connection to calculate the default rate and the average rate of recovery \(P_1, R_1\), then use the new issuer and the following issuer who has the largest correlation coefficient to carry out the second two normal Copula connection, and so on, finally get \(P_{n-1}, R_{n-1}\). The specific process is as follows:

a. Assuming under the circumstance that a single default probability \(P_n\) and recovery rate \(R\) (assuming each issuer has the same recovery rate) of each issuer at time \(t\) are known, we start conducting the first connection by merging the two issuers (assuming they are A, B) who have the largest correlation coefficient. In two normal Copula connection structure, the probability \(P_{AB}\) of both A, B default can be expressed as:

\[
C(P(A), P(B); \rho_{AB}) = \int_{-\infty}^{\Phi^{-1}(P(A))} \int_{-\infty}^{\Phi^{-1}(P(B))} \frac{1}{2\pi \sqrt{1-\rho_{AB}^2}} \exp \left[ -\frac{(r^2 + s^2 - 2\rho_{AB}rs)}{2(1-\rho_{AB}^2)} \right] dr ds
\]  \(\text{(8)}\)

At this time point, the default probability of one of A, B at least is: \(P = P_A + P_B - P_{AB}\)

In the case of one of A and B at least is default, the average recovery rate is at least:
\[ R_1 = (1 - P_1) \times 1 + (P_A - P_{AB}) \times \frac{w_A}{w_A + w_B} \times R + (P_B - P_{AB}) \times \frac{w_B}{w_A + w_B} \times R + P_{AB} \times R \]  

(9)

\( w_A, w_B \) represent respectively A and B in the proportion of total debt in the collective bonds.

b. According to the correlation coefficient matrix \( \rho \), we calculate respectively the correlation coefficient between the first connection and the rest default issuers:

\[ \rho_{1,C} = \frac{\rho_{AC} + \rho_{BC}}{2}, \rho_{1,D} = \frac{\rho_{AD} + \rho_{BD}}{2}, \ldots. \]

Choosing the one whose correlation coefficient is the largest (assuming is C) to be the second connection. In two normal Copula connection structure, the probability of both of them default \( P_{1.C} \) can be expressed as:

\[ C(P_1, P(C); \rho_{1.C}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi \sqrt{1-\rho_{1.C}^2}} \exp\left(-\frac{(r^2 + s^2 - 2\rho_{1.C}rs)}{2(1-\rho_{1.C}^2)}\right) dr ds \]  

(10)

At this time point, the default probability of one of A, B, C at least is: \( P_2 = P_1 + P_c - P_{1.C} \).

In the case of one of A, B and C at least is default, the average recovery rate is at least:

\[ R_2 = (1 - P_2) \times 1 + (P_1 - P_{1.C}) \times \frac{w_1}{w_1 + w_c} \times R + (P_c - P_{1.C}) \times \frac{w_c}{w_1 + w_c} \times R + P_{1.C} \times R \]  

(11)

\( w_1, w_c \) represent respectively the first connection and C in the proportion of total debt in the collective bonds.

According to the principle of step 1, 2, by the cycle of calculation, we constantly connect the new default issuer, and finally we can get \( P_{n-1}, R_{n-1} \), which represent the probability of at least one of the issuers default and average recovery rate when default happens.

### 3.3 Calculation of credit spreads and the processing of embedded options

#### 3.3.1 Calculation of credit spreads

By the above method, we can calculate the cumulative default rate \( P_t \) and the average recovery rate \( R_t \) of the collective bonds at any time \( t \) after issuing. At this time we can use the risk neutral pricing formula to calculate the credit spread of the collective bonds. Based on the above discussion of the recovery rate and the probability of default, we can get the following credit spreads:

\[ \sum_{i=1}^{n} \left[ \frac{C}{(1 + r_j + CS)} + \frac{F}{(1 + r_j + CS)} \right] = \frac{P_t \times R_t \times F}{1 + r_f} + \sum_{i=2}^{n} \left[ (P_t - P_{t-1}) \left( \frac{C}{(1 + r_j)^{i-1}} \right) + \frac{R_t \times F}{(1 + r_j)^i} \right] \]  

Where \( C \) is interest, \( F \) is par value, \( r_f \) is risk-free interest rate, \( CS \) is credit spreads.

#### 3.3.2 The processing of embedded options

At present, it is common to set put option clause in collective bonds, so put option must be considered in the pricing. In practice, the BDT model is close to the industry standard, which is suitable for the pricing of bonds and general interest rate derivatives. On the assumption that the upward shift probability of short-term interest rates each is 0.5, continuous time expression of BDT model is:

\[ d \ln r(t) = \left[ \theta(t) - \frac{\sigma'(t)}{\sigma(t)} \right] dt + \sigma(t) dw \]  

Where \( \theta(t) \) and \( \sigma(t) \) are the parameters of time, and can be used to fit observed market interest rate and interest rate volatility.

Finally, according to the formula (8) and the credit spread which is obtained by the last part, we can complete the construction of the two fork trees, and complete the pricing of the collective bonds with embedded options.
4. Conclusion

The Copula function introduced in this paper provides a new technical route for the measurement of the joint breach between different issuers. On the basis of the calculation of the credit spreads of the collective bonds, the pricing method of the small and medium enterprise collective bonds is improved, which helps to control the credit risk more accurately and promote the development of the market.

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References


