Newton’s Method of Wind Turbine Flow Field Numerical Simulation of Local Convergence Analysis

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Abstract. In this paper, we design a modeling analysis method around the wind turbine flow field. First, a mathematical model of the cylinder coordinate system is proposed to describe the model of wind turbine flow field. The finite difference method and Newton iteration method are used to solve numerically, and the local convergence conditions are given to local convergence analysis of Newton iterative format. This condition not only gives the time step, space interval relations and parameters, but also provides a theoretical basis for the numerical solution of flow convergence of Newton’s method for wind turbine. Simulation results show that the actual value of time step calculated in the direct decoupling method is larger than the theoretical value.

Introduction

The research on the theory of wind turbine belongs to the category of low-speed aerodynamics [1-3]. The diameter of large wind machine is larger, and the change of wind condition is more complicated in the natural environment [4]. Around the wind turbine flow field is compressible, three-dimensional unsteady viscous flow field, the mathematical model of which is composed of multiple variables which are unknown functions of nonlinear, partial differential equations of the unsteady definite solution problem [1,5]. Currently, the domestic research on aerodynamic characteristics analysis of the wind turbine itself mainly reflected in two aspects: One is the blade optimization design [6], and the second is wind generator system control, and energy conversion process; The wind turbine aerodynamic flow field simulation is mainly done in a certain degree of simplification for the mathematical model of wind turbine, and then using indirect coupled method for numerical solution [7]. Because of the difficulties of the convergence of this method and the slow pace of the convergence, using the direct coupling method in flow field numerical wind turbine issue is of great theoretical significance and application value [8].

The research of this paper is without considering effects of blade tip and blade tip part and the characteristics of turbine airfoils under the premise of appropriate simplification. The wind turbine flow field of solving regional is regarded as cylindrical in shape from the dynamic rotating blades, and a mathematic model of the cylindrical coordinates is built on it. Limited difference method and Newton method of combined are applied to wind machine flow field numerical simulation. In this research, Newton iterative format local convergence is theoretically analyzed, local convergence conditions are given to provide the theoretical basis
for the selection of suitable space step, and the conditions of the local convergence of Newton iteration are verified through a numerical example.

**The Definite Condition of Solution Area**

![Figure 1. Solving region \( \Omega \).](image)

The solving area of air cylinder flow field around the wind turbine includes the entrance and exit parts (Eq. 1), among

\[
\begin{align*}
\Omega_1 &= \{(x, r, \theta) \mid X_1 \leq x < 0, 0 \leq r \leq R_1, 0 \leq \theta \leq 2\pi\} \\
\Omega_2 &= \{(x, r, \theta) \mid 0 < x \leq X_2, 0 \leq r \leq R_1, 0 \leq \theta \leq 2\pi\} \\
\end{align*}
\]

boundary \( \Gamma_1 \) is the outer surface section of area \( \Omega \) and \( \Gamma_2 \) is the interface of \( \Omega_1 \) and \( \Omega_2 \). \( \Gamma_3 \) is the definite condition (Eq. 2) of control equation under columnar coordinate system

\[
\begin{align*}
\Gamma_3 &= \left\{(x, r, \theta) \mid x = 0; r \leq R_0; \omega t + \frac{2k \pi}{3} - \frac{\alpha}{2} \leq \theta \leq \omega t + \frac{2k \pi}{3} + \frac{\alpha}{2}, k = 0, 1, 2\right\}
\end{align*}
\]

among: The wind wheel radius is \( R_0 \), Wind string length is \( C \), \( \alpha = \arcsin \frac{C}{R_0} \). The controller can change the aerodynamic characteristics of wind turbines by adjusting the blade pitch angle \( \beta \) to stable wind speed and wind turbine output power when the wind speed value is larger.

**Discrete Formats and Newton Iteration**

As is shown in Fig. 1 of solving regional subdivision:

\[
G = \{x_i, r_j, \theta_k \mid x_i = X_1 + ih_x, i = 0, 1, \ldots, N_x + 1; \\
r_j = (j + 0.5)h_r, j = 0, 1, \ldots, N_r; \theta_k = kh_{\theta}, k = 1, \ldots, N_{\theta}\}
\]

among: \( h_x = \frac{X_2 - X_1}{N_x + 1} \), \( h_r = \frac{R_1}{N_r} \).
\[ h_\theta = \frac{2\pi}{N_\theta} \]

Node \((x_i, r_j, \theta_k)\) is shorthand for \((i, j, k)\). Take step \(\tau\), time \(t_m = m\tau (m = 1, 2, \cdots)\). Backward differencing scheme in time derivative format of space derivative term using central difference scheme is introduced into Eq.1 at the moment, the node in the space. The discrete nonlinear equations format (Eq.3) of Eq.1 are as follows

\[ F(X^{(m)}) = 0 \]  (3)

among:
\[ X^{(m)} = (\mu_{i11}^m, v_{i11}^m, w_{i11}^m, \rho_{i11}^m, \cdots, w_{N_N, N_\theta}^m, \rho_{N_N, N_\theta}^m)^T \]

By using Newton iterative method for solving Eq.8, Newton’s format (Eq.4)

\[ F'(X^{(m)}) \Delta X^{(m)} = -F(X^{(m)})^{(l)} \]

\[ X^{(m)}_{(l+1)} = X^{(m)}_{(l)} + \Delta X^{(m)}_{(l)}, \quad l = 0, 1, \cdots \]  (4)

among, \(l\) is the number of iterations, the vectors of a time layer \(X^{(m-l)}\) is used as the approximate solution of initial vector \([X^{(m)}]^{(0)}\). To solve the equation with the super relaxation method.

**Analysis of Local Convergence of Newton’s Method**

Local convergence analysis is one of the important criteria for judging a numerical method. According to the literature [3], there are the following lemmas:

**Reference theorem 1**
Supposing that \(A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}\) is the strictly diagonally dominant matrix, \(A\) is nonsingular matrix.

**Reference theorem 2**
Supposing that \(X^* \in \Omega\) is the solution of nonlinear equation group \(F(X) = 0, \ X^* \in \Omega\). \(F(X)\) satisfies the following condition:

1. \(F(X)\) is continuously differentiable in the area of \(S(X^*, \delta) \subseteq \Omega\), and \([F(X^*)]^{-1}\) exists;
2. \(F(X)\) is second order differentiable in \(X^*\), and \(F'(X^*)h \neq 0, \forall h \in \mathbb{R}^n, h \neq 0\).

Then get the Newton iteration (Eq.5) of \(F(X) = 0\)

\[
\begin{aligned}
X^{(l+1)}_{(l)} &= X^{(l)}_{(l)} + \Delta X^{(l)}_{(l)} \\
F'(X^{(l)}_{(l)})\Delta X^{(l)}_{(l)} + F(X^{(l)}_{(l)}) &= 0, \quad l = 0, 1, 2, \cdots
\end{aligned}
\]  (5)

Get the following theorem by the lemma above.

**Theorem**
If the time step meets \(\tau \leq \min\{\tau_1, \tau_2, \tau_3\}\), The Newton iteration is local second-order convergence.
\[ \tau_1 = C_2 \left[ 3A_1C_1^2 + \frac{2\mu}{h_x^2} + \frac{RT}{h_x} + A_2 \frac{\mu}{3h_x} \right]^{-1} \]

\[ \tau_2 = C_2 \left[ 3A_1C_1^2 + A_2 \left( RT + \frac{\mu}{3h_x} + \frac{2\mu}{h_r^2} \right) \right]^{-1} \]

\[ \tau_3 = \left[ 2A_1C_1 \right]^{-1} \]

\[ A_1 = \frac{1}{h_x} + \frac{2}{h_r} + \frac{2}{h_\theta}, \quad A_2 = \frac{1}{h_r} + \frac{1}{h_\theta} \]

\[ C_1 = \| X^* \|_\infty \quad (X^* \text{ is the solution}) \]

\( C_2 \) represents the minimum value of density \( \rho \) in time \( t_{m-1} \) in the area \( \Omega \).

\( F'(X^*) \) is strictly diagonally dominant in the line \( l \). To make row \( l+1 \) of \( F'(X^*) \) strictly diagonally dominant, only the time step \( \tau \) meets

\[ \tau < C_2 \left[ \frac{3C_1^2}{h_x} + 3C_1^2 \left( \frac{|\cos \theta_k|}{h_r} + \frac{|\sin \theta_k|}{r_j h_\theta} + \frac{1}{h_\theta} \right) + 3C_1^2 \left( \frac{|\sin \theta_k|}{h_r} + \frac{|\cos \theta_k|}{r_j h_\theta} + \frac{1}{h_\theta} \right) + RT \left( \frac{|\cos \theta_k|}{h_r} + \frac{|\sin \theta_k|}{r_j h_\theta} + \frac{1}{h_\theta} \right) + \frac{\mu}{3h_x^2} \left( \frac{|\sin 2\theta_k|}{r_j h_\theta} + \frac{1}{h_\theta} \right) + \frac{2\mu}{r_j h_\theta} \left( \frac{|\sin 2\theta_k|}{h_r^2} + \frac{|\sin 2\theta_k|}{h_\theta^2} + \frac{1}{r_j h_r^2} \right) + \frac{2\mu}{h_r} \left( \frac{|\sin \theta_k|}{h_\theta^2} \right) \right]^{-1} \]

Among: \( X^* \) is the solution of Newton iterative format. Take \( \| X^* \|_\infty = C_1 \), \( C_2 \) represents the minimum value of density \( \rho \) in time \( t_{m-1} \) in the area \( \Omega \).

That is, when the time step \( \tau \) meets:

\[ \tau \leq C_2 \left[ 3A_1C_1^2 + A_2 \left( RT + \frac{\mu}{3h_x} + \frac{2\mu}{h_r^2} \right) \right]^{-1} = \tau_2 \]

\( F'(X^*) \) is strictly diagonally dominant in the line \( l+1 \).

Similarly, when time step \( \tau \) meets:

\[ \tau \leq C_2 \left[ 3A_1C_1^2 + A_2 \left( RT + \frac{2\mu}{3h_x} + \frac{\mu}{3h_r} + \frac{\mu}{3h_\theta} \right) \right]^{-1} = \tau_2 \]

\( F'(X^*) \) is strictly diagonally dominant in the line \( l+2 \).

In the same way, when the time step \( \tau \) meets:

\[ \tau \leq \left[ 2A_1C_1 \right]^{-1} = \tau_3 \]

\( F'(X^*) \) is strictly diagonally dominant in the first \( l+3 \) line.

Therefore, when \( \tau \leq \min\{ \tau_1, \tau_2, \tau_3 \}, F'(X^*) \) is strictly diagonally dominant, \( [F'(X^*)]^{-1} \) exists.

secondly, prove \( F(X^{(m)}) \) meets the conditions of lemma 2. Obviously, \( F(X) \) is second order
differentiable in $X^*$. In the same way, the first $l+1$, $l+2$, $l+3$ component of $F''(X^*)hh$ is non-zero. So we have: $F''(X^*)hh \neq 0$. In summary, according to the lemma, Newton iterative scheme (9) are local second-order convergence.

**Numerical Experiment**

In order to verify the validity of the condition and viability of local convergence of Newton's method according to the paper. A computer program of the concrete calculating examples are numerically simulated corresponding FORTRAN language. Parameter values are as follows: $g=9.8m/s^2$, $A=\pi r_1^2$, $a=1/3$, $\rho_0=1kg/m^3$, $\omega=1.6rad/s$, $R=300J/mol*k$, $T=288K$, $\mu=0.000018$, $v_0=15m/s$. The following three situations are simulated step by step.

(1)$X_1=-1.00m$, $X_2=1.00m$, $R_1=1.00m$, $R_0=0.50m$, $C=0.02m$, $N_x=33$, $N_r=33$, $N_\theta=35$; For different time $\tau$, the convergence of Newton's method is shown in Table 1.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$N_x$</th>
<th>$N_r$</th>
<th>$N_\theta$</th>
<th>status</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>33</td>
<td>33</td>
<td>35</td>
<td>$m=1$ divergence</td>
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<td>0.0005</td>
<td>33</td>
<td>33</td>
<td>35</td>
<td>$m&gt;1000$</td>
</tr>
</tbody>
</table>

(2)$X_1=-1.00m,X_2=1.00m,R_1=1.00m,R_0=0.50m,C=0.02m,N_x=63$, $N_r=63$, $N_\theta=65$; For different time $\tau$, the convergence of Newton's method is shown in Table 2.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$N_x$</th>
<th>$N_r$</th>
<th>$N_\theta$</th>
<th>status</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>63</td>
<td>63</td>
<td>64</td>
<td>$m=1$ divergence</td>
</tr>
<tr>
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<td>63</td>
<td>64</td>
<td>$m=5$ divergence</td>
</tr>
<tr>
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<td>63</td>
<td>63</td>
<td>64</td>
<td>$m&gt;1000$</td>
</tr>
</tbody>
</table>

(3)$X_1=-10.00m,X_2=10.00m,R_1=5.00m,R_0=2.50m,C=0.20m,N_x=63,N_r=63$, $N_\theta=64$; For different time steps $\tau$, The convergence of Newton's method is shown in Table 3.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$N_x$</th>
<th>$N_r$</th>
<th>$N_\theta$</th>
<th>Convergence condition</th>
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<tr>
<td>0.0005</td>
<td>63</td>
<td>63</td>
<td>64</td>
<td>$m&gt;1000$</td>
</tr>
</tbody>
</table>

We can see from Table 1 to Table 3: (1) Convergence of Newton's method is associated
with the time step value; (2) With the reduced space and reduces, time step of convergence of Newton's method reduces; (3) When the time step $\tau$ is up to a certain value $\tau = 0.0002$, Newton's method converges meet in time layer $n > 1000$, theorem of time step should be met $\tau < 0.00001$, which suggests that time step $\tau$ actual value is larger than the theoretical value.

Conclusions

(1) Wind turbine flow field under unsteady nonlinear partial differential equations boundary value problems is established based on the cylindrical shape and leaf dynamics of rotating, cylindrical coordinate system.

(2) Combine the finite difference method with Newton's method for numerical solution, giving the finite difference method for discrete format with the Newton iteration.

(3) To analyze the local convergence of the Newton iteration in theory, giving local convergence conditions which give the time step and space interval relations and parameters, and provide theoretical basis for selecting the appropriate time step.

(4) The local convergence of Newton iteration criteria is verified through a numerical example, which also show that actual time $\tau$ value is larger than the theoretical value.

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References


