Dynamic Analysis and Linear Feedback Control of a Three-Dimensional Energy Price System

Xing-hua FAN, Ying ZHANG and Jiu-li YIN

Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China

*Corresponding author

Keywords: Energy Price dynamic system, Chaos, Chaotic attractor, Equilibrium point, Linear feedback control.

Abstract. Evolution of energy price presents new characteristics in the time of carbon emissions trading. This study sets up a new three-dimensional Energy Price chaotic system based on the interdependent and mutually restricted evolution relationship among carbon price, energy price, and economic growth during a given economic period. The dynamic behavior of the system is studied. All equilibrium points are unstable for the system under certain parameters. Lyapunov exponents spectrum and bifurcation diagram present the existence of chaos, and numerical simulation shows the chaotic behavior of the dissipative system. Then the linear feedback control method is used to control the system to the original unstable equilibrium point. Numerical simulations verify the correctness of the theoretical analysis.

Introduction

Fossil energy consumption are the main contributor for greenhouse gas emissions and causes global warming. In order to control greenhouse gas emissions, the Kyoto Protocol came into force in 2005, with the aim of promoting countries to reduce carbon dioxide emissions. Under the Protocol, carbon emission market was set up and the carbon price has attracted widespread attention. Furthermore, in order to effectively control carbon emission reduction while simultaneously maintaining a healthy rate of economic development, some scholars have suggested that energy prices are the most basic and effective ways to determine resource allocation, balance energy conservation and carbon emissions [1].

Energy prices are firmly related to carbon price. Energy price is the major factor impacting carbon price volatility in the EU market of the first phase [2]. There are bi-directional and time-varying spillovers between carbon and energy markets [3]. Study shows a stronger energy price impacts on EUA prices than CER prices and a positive energy price impact on EUA prices [4].

There is an interaction between energy price and economic growth. In order to find out whether energy price regulation would benefit China’s economy, Ju et al. [5] conducted an in-depth simulation using path analysis. It was found that absolute distortions of energy prices have negative impacts on economic growth. Mahadevan [6] provided us with more in-depth work to analyze energy prices, energy consumption and economic growth by statistical data and the panel error correction model. Arshad et al. [7] empirically explored the impact of energy prices on economic growth by using GMM estimation technique. The results showed that the overall impact of energy prices on growth was negative, and revealed that most of the impact of energy price index on economic growth was reflected in stock prices, real exchange rate, government consumption, and unemployment.

The previous researches mainly discussed the impact of energy price on carbon price, economic growth, and other factors using econometric or statistical methods. Due to the complexity of the energy market and carbon market, the interactions across energy price, carbon price, and economic growth present complicated characteristics. This paper uses a nonlinear dynamical system to clarify the interaction between prices and to reflect nonlinearity well. Nonlinear dynamic systems typically use nonlinear equations to describe the behavior of complex dynamic systems. Chaos analysis and
applications in dynamical systems have been observed in many practical applications in engineering and economics [8,9].

Chaos is a seemingly random irregular movement. It is an inherent property of nonlinear dynamic systems, especially highly sensitive and unpredictable for small changes in initial conditions. Controlling chaotic states to stable states has practical significance [8,10]. Many chaos control methods have been proposed [11], among which the linear feedback control method has obtained special interesting [12]. In this paper, a linear feedback control method is used to control the three-dimensional Energy Price chaotic system to the original unstable equilibrium point, and a new three-dimensional Energy Price stabilization system is obtained.

The purpose of this paper is to study the complex behavior of energy price in a low-carbon background. For this aim, theory and method of the nonlinear dynamic system are applied. We build a three-dimensional chaotic system about carbon price, energy price, and economic growth based on their complex relationships. Then we locate the conditions for chaos by dissipative analysis and parameter analysis. The existence of chaos is verified by Lyapunov exponents spectrum and bifurcation diagram. Finally, a stable behavior of the Energy Price system is realized by the linear feedback control method. Numerical simulations verify the correctness of the results.

**Establishment of the Energy Price Dynamic System**

Combined with the interdependent and mutually constrained evolutionary relationship between carbon price, energy price, and economic growth during a given economic period, a three-dimensional Energy Price dynamic evolution model is obtained as follows:

\[
\begin{align*}
\dot{x} &= a_1 x \left(\frac{z}{M} - 1\right) + a_2 y - a_3 z, \\
\dot{y} &= b_1 y - b_2 x + b_3 z \left(\frac{z}{L} - 1\right), \\
\dot{z} &= -c_1 x + c_2 y + c_3 z \left(1 - \frac{z}{N}\right),
\end{align*}
\]

where \(x(t)\) is the time-dependent variable of carbon price, \(y(t)\), of energy price, \(z(t)\), of economic growth; \(a_i, b_i, c_i, (i = 1, 2, 3), M, L, N\) are positive constants, \(t \in I, I\) is a given economic period.

The first sub-equation of Eq. 1 expresses the complicated relationship between carbon price, its rate of change, energy price, and economic growth during a given period. It indicates that the rate of change of carbon price \(dx(t)/dt\) is associated with carbon price, \(x(t)\), and the share of carbon price potential, \((z/M - 1)\), simultaneously in a positive proportional to them. As for \(a_1 x(z/M - 1)\), when \(z < M\), the carbon price decrease in a developing economic period; when \(z > M\), the carbon price increase in a developed economic period. The rate of change of carbon price is positively proportional to energy price and negatively proportional to economic growth.

The second sub-equation of Eq. 1 indicates that the rate of change of energy price \(dy(t)/dt\) is negatively proportional to carbon price. In its developing period, economic growth counteracts the growth of energy price, the influence of which on energy price becomes stronger after the peak value \(L\). As for \(b_2 z(z/L - 1)\), when \(z < L\), the influence of economic growth on energy price is negative; when economic growth arrives at \(L\), the influence of economic growth on energy price is positive.

In the third sub-equation of Eq. 1, the rate of change of economic growth \(dz(t)/dt\) is negatively proportional to carbon price. This rate is positively proportional to energy price. The early development level of economic growth is high and gradually decreases after the threshold \(N\). As for \(c_3 z(1 - z/N)\), when \(z < N\), the economic growth increase in its developing period; when \(z > N\), the economic growth decrease in its developed period.
Dynamic Analysis of the Energy Price System

We discuss the dynamic characteristics of the system using theoretical analysis and numerical simulation.

Equilibrium Point Analysis

Eq. 1 has three equilibriums: \(O(0,0,0)\), \(S_1(x_1, y_1, z_1)\), and \(S_2(x_2, y_2, z_2)\). We see that the first equilibrium point is always the origin while the other two points are determined by the parameters.

We identify the type of equilibrium points by eigenvalues. The Jacobian matrix at the equilibrium point \(O(0,0,0)\) is

\[
J_0 = \begin{pmatrix}
-a_1 & a_2 & -a_3 \\
-b_2 & b_1 & -b_3 \\
-c_1 & c_2 & c_3
\end{pmatrix}.
\] (2)

We determine the parameter set when the equilibrium point \(O(0,0,0)\) is unstable. For the following fixed parameters except for \(b_3\),

\[
a_1 = 0.048, a_2 = 0.012, a_3 = 0.001, b_1 = 0.02, b_2 = 0.024, c_1 = 0.012, \]
\[
c_2 = 0.013, c_3 = 0.008, L = 1.7, M = 0.9, N = 0.3.
\] (3)

The characteristic polynomial of \(J_0\) is

\[
f(\lambda) = \lambda^3 + 0.02\lambda^2 + (0.013b_3 - 0.000908)\lambda + (0.000005304 + 0.00048b_3) = 0.
\] (4)

Let \(p_1 = 0.02, p_2 = 0.013b_3 - 0.000908,\) and \(p_3 = 0.000005304 + 0.00048b_3\). When \(b_3 < -0.10665\), we have \(p_1 > 0, p_2 > 0, p_1p_2 - p_3 > 0\). By the Routh-Hurwitz criterion, all real eigenvalues and all real parts of complex conjugate eigenvalues of Eq. 4 are negative. For example, we let \(b_3 = 0.12\), the parameters are shown in Eq. 5:

\[
a_1 = 0.048, a_2 = 0.012, a_3 = 0.001, b_1 = 0.02, b_2 = 0.024, b_3 = 0.12, \]
\[
c_1 = 0.012, c_2 = 0.013, c_3 = 0.008, L = 1.7, M = 0.9, N = 0.3.
\] (5)

The eigenvalues at \(O(0,0,0)\) are \(\lambda_1 = -0.0412, \lambda_{2,3} = 0.0160 \pm 0.0376i\). Therefore, for any positive constant \(b_3\), the equilibrium point \(O(0,0,0)\) is unstable.

Fix parameters as shown in Eq. 5, the equilibrium points are \(S_1(-6.8017, -5.6811, 0.7101)\) and \(S_2(-0.9390, 1.1163, 1.1447)\). The eigenvalues at \(S_1\) are \(\lambda_1 = -0.0740, \lambda_2 = 0.0594, \lambda_3 = -0.0054\); at \(S_2\) are \(\lambda_1 = -0.0652, \lambda_{2,3} = 0.0226 \pm 0.0089i\). Therefore, both \(S_1\) and \(S_2\) are saddle points.

Dissipation of the Dynamic System

The divergence of Eq. 1 is

\[
\nabla V = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z} = a_i z + a_i b_i + c_i \frac{2c_i z}{N} = \left(\frac{a_1}{M} - \frac{2c_3}{N}\right) z - a_i + b_i + c_i.
\] (6)

If \(a_i = \frac{2c_3}{N}\), \(-a_i + b_i + c_i < 0\), then the dynamic system presented in Eq. 1 is a dissipative system.

Numerical Simulation

A dissipative system with unstable saddle points may lead to chaos. We show the existence of chaos through numerical simulation.
We chose a set of parameters as shown in Eq. 5. Let initial condition be \([0.0169, 0.2, 0.3]\). A chaotic attractor is observed (Fig. 1(a)). The corresponding chaotic time series are shown in Fig. 1(b).

![Energy Price chaotic attractor](image1.png)

**Figure 1.** Energy Price chaotic attractor. (a) 3D phase diagram. (b) Time series.

The existence of chaos is also checked by Lyapunov exponents spectrum and bifurcation diagram. Vary \(b_3\) and fix parameters as shown in Eq. 3. The Lyapunov exponents spectrum of parameter \(b_3\) is shown in Fig. 2. The bifurcation diagram with respect to parameter \(b_3\) is shown in Fig. 3.

![Lyapunov exponent spectrum](image2.png)

**Figure 2.** Lyapunov exponent spectrum.

![Bifurcation diagram](image3.png)

**Figure 3.** Bifurcation diagram of \(z\) for parameter \(b_3\).

According to the Lyapunov exponents and bifurcation diagram of \(b_3\), the sensitivity of the system to the parameter can be seen, as shown in Fig. 2 and Fig. 3. The largest Lyapunov exponent in Fig. 2 is greater than zero, which indicates that the system presented in Eq. 1 has a chaotic attractor as shown in Fig. 1. Besides, the bifurcation in Fig. 3 indicates that the system is unstable.

**Linear Feedback Control of Energy Price System**

In this section, the linear feedback control method [13] is applied to control chaos in the three-dimensional Energy Price system.

Firstly, we prove the chaotic system can be controlled to the equilibrium point \(O(0,0,0)\). We guide the chaotic trajectory \((x(t), y(t), z(t))\) to the equilibrium point \(O(0,0,0)\). Let the system presented in Eq. 1 be controlled by a linear feedback rule:

\[
\begin{align*}
\dot{x} &= a_1 x \left(\frac{z}{M} - 1\right) + a_2 y - a_3 z - F_{11} x, \\
\dot{y} &= b_1 y - b_2 x + b_3 z \left(\frac{z}{L} - 1\right) - F_{22} y, \\
\dot{z} &= -c_1 x + c_2 y + c_3 z \left(1 - \frac{z}{N}\right) - F_{33} z,
\end{align*}
\]

where \(F_{11}, F_{22}, F_{33}\) are positive feedback gains, which are needed to be chosen such that the trajectory of the system presented in Eq. 1 is stabilized to the equilibrium point \(O(0,0,0)\).

We will prove all the real parts of the eigenvalues of the controlled system are negative. For the parameters as shown in Eq. 5, the Jacobian matrix of Eq. 7 is
\[
J_0 = \begin{pmatrix}
-0.048 - F_{11} & 0.012 & -0.001 \\
-0.024 & 0.02 - F_{22} & -0.12 \\
-0.012 & 0.013 & 0.008 - F_{33}
\end{pmatrix}.
\] (8)

Let \( F_{11} = F_{33} = 0 \). Eq. 8 has the characteristic polynomial

\[
f(\lambda) = \lambda^3 + (0.02 + F_{22})\lambda^2 + (0.04F_{22} + 0.000652)\lambda + (0.000062904 - 0.000396F_{22}) = 0.
\] (9)

According to the Routh-Hurwitz criteria, if

\[
\begin{align*}
0.02 + F_{22} &> 0, \\
0.000062904 - 0.000396F_{22} &> 0, \\
(0.02 + F_{22})(0.04F_{22} + 0.000652) &> 0.000062904 - 0.000396F_{22},
\end{align*}
\] (10)

is true, then the Jacobian matrix \( J_0 \) has three negative real part eigenvalues. When \( F_{22} \) satisfies Eq. 10, the controlled system presented in Eq. 7 is asymptotically stable at the equilibrium \( O(0,0,0) \).

Numerical simulation confirms the above theoretic analysis. Numerical experiments are carried out to integrate the controlled system presented in Eq. 7 by the MATLAB. The parameters are chosen as shown in Eq. 5 to ensure the existence of chaos in the absence of control. Let initial states be \([0.0169, 0.2, 0.3]\). When \( F_{11} = F_{33} = 0, F_{22} = 0.03 \), the equilibrium point \( O(0,0,0) \) of the system presented in Eq. 7 is stabilized as shown Fig. 4.

**Figure 4.** Controlling system 8 to the origin. (a) 3D phase diagram. (b) Time series.

**Summary**

Interaction of carbon price, energy price, and economic growth is modeled by a three-dimensional dynamic system in this paper. The constructed dynamical system presents complex behavior. By equilibrium stability analysis and dissipative analysis, this study finds a possible parameter set for the chaos. Under such a parameter set, the system is proved to be chaotic by the Lyapunov exponents spectrum and bifurcation diagram. To avoid chaos in the energy market, the linear feedback control method is applied to control the chaos. By adding the feedback on the variable of energy, the system is controlled to the origin for a continuous interval of the control parameters.

This study demonstrates a successful application of nonlinear dynamic system theory. Although energy price has been engaged in some nonlinear dynamic systems, such as energy prices-energy efficiency-economic growth dynamic system and energy prices–energy supply–economic growth dynamic system, it is a new attempt to consider both the carbon market and energy market and take their prices as system variables. The model and the results of this study may help further researches, such as identifying an experimental Energy Price system, scenario analysis for different system parameters, or new control methods for the Energy Price system.

**Acknowledgement**

This research was financially supported by the National Natural Science Foundation of China (No. 71673116) and the Humanistic and Social Science Foundation from Ministry of Education of China (Grant 16YJAZH007).
References


