Sensitivity Function with Applications to Analyze Frequency Characteristics under Conditions of System Parameter Varying

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Abstract. This paper focuses on analyzing the sensitivity of frequency characteristics of closed-loop system under reference input and external disturbance. In the first place, the sensitivity function of linear time-invariant control system is defined. Then the sensitivity functions of frequency characteristics including plant and controller parameters are derived in accordance with the closed-loop transfer functions and error transfer functions of closed-loop systems. And then the sensitivity of amplitude-frequency characteristics, while plant parameter and controller parameter are varying, is analyzed. Application results indicate the effects, which the plant parameter and controller parameter exert on the frequency characteristics of closed-loop systems, can be analyzed directly by sensitivity function. The analysis method about frequency characteristics facilitates the plant parameter design and the controller parameter determination for closed-loop systems.

Introduction

The closed-loop transfer function and the error transfer function are both often used model for analyzing and designing feedback control systems. In order to analyze or design the feedback control system, correspondingly, both the closed-loop frequency characteristic and the error frequency characteristic have to be considered [1, 2]. Because of combined actions of the reference input and the disturbance input, it is not easy to analyze or design the closed-loop and error frequency characteristics directly [3, 4]. Furthermore, when the design of controlled plant parameters or the selection of controller parameters are taken into account, the analyses on closed-loop and error frequency characteristics will be more complicated and interactive [5].

In this paper, the closed-loop and error amplitude-frequency characteristics under the conditions of plant or controller parameter varying are compared and analyzed by the sensitivity functions [6] of above amplitude-frequency characteristics. Firstly, the sensitivity function is defined and the sensitivity functions of the closed-loop and the error amplitude-frequency characteristics are derived. And then these sensitivity functions are applied to analyze the effects on closed-loop and error amplitude-frequency characteristics while the plant or controller parameters are varying.

Sensitivity Function and Its Applications to Closed-Loop Control Systems

Definition of Sensitivity Function

Considering a control system, one of its characteristics is denoted by $W$ and $W$ is affected by some one of parameters, $a$. The system characteristic $W$ is changed when system parameter $a$ varies. The system characteristic changing with system parameter variation is called sensitivity. Considering the ratio of the percent change in the system characteristic $W$ to the percent variation in the system parameter $a$ and denoting this ratio as $S$

$$S = \frac{\Delta W / W}{\Delta a / a} = \frac{\Delta W}{\Delta a} \frac{a}{W},$$

(1)
where $\Delta W$ is the change in the system characteristic $W$ due to the variation $\Delta a$ in the system parameter $a$. Evaluating mathematically this sensitivity, it is defined as the sensitivity function of system characteristic about system parameter, and represented by the limit form as $\Delta a$ approaches zero. Hence, the sensitivity function is given by

$$S_a^W = \lim_{\Delta a \to 0} \frac{\Delta W}{\Delta a} = \frac{\partial W}{\partial a}.$$

(2)

For a linear time-invariant control system, transfer function is its most important characteristic. Thus, the sensitivity function is defined as

$$S_a^W(s) = \frac{\partial W(s)}{\partial a} \frac{a}{W(s)},$$

(3)

where $s$ is the Laplace transform variable factor. In general, this sensitivity function is also a function of $s$, which makes the sensitivity function difficult to interpret. Associating it with the frequency characteristic of the control system, i.e. letting $s = j\omega$, there yields the sensitivity function of frequency characteristic

$$S_a^W(j\omega) = \frac{\partial W(j\omega)}{\partial a} \frac{a}{W(j\omega)}.$$

(4)

**Sensitivity Function of Closed-Loop Control System**

Giving a typical closed-loop control system, its control block diagram is shown as Figure 1, where $G_p(s)$ is the transfer function of controlled plant with varying parameter $a$, $G_c(s)$ is the transfer function of controller with varying parameter $b$, and $H(s)$ is the transfer function of feedback channel. And $R(s)$, $C(s)$, $B(s)$, $E(s)$, and $D(s)$ are the reference input, controlled output, feedback variable, error variable, and disturbance variable in complex domain, respectively.

![Figure 1. Control block diagram of typical closed-loop system.](image)

In accordance with Figure 1, the closed-loop transfer function from reference input $R(s)$, $\Phi_R(s)$, can be derived as

$$\Phi_R(s) = \frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)H(s)},$$

(5)

the closed-loop transfer function from disturbance variable $D(s)$, $\Phi_D(s)$, can be derived as

$$\Phi_D(s) = \frac{C(s)}{D(s)} = \frac{G_p(s)}{1+G_c(s)G_p(s)H(s)},$$

(6)

the error transfer function from reference input $R(s)$, $\Phi_{RE}(s)$, can be derived as

$$\Phi_{RE}(s) = \frac{E(s)}{R(s)} = \frac{1}{1+G_c(s)G_p(s)H(s)},$$

(7)

and the error transfer function from disturbance variable $D(s)$, $\Phi_{DE}(s)$, can be derived as

$$\Phi_{DE}(s) = \frac{E(s)}{D(s)} = \frac{-G_p(s)H(s)}{1+G_c(s)G_p(s)H(s)}.$$
According to the definition (3), sensitivity functions of closed-loop transfer functions and error transfer functions about plant parameter a and controller parameter b can be derived as

\[
\begin{align*}
S_{\phi a}^p(s) &= \frac{\partial \Phi_a(s)}{\partial a} \phi_a(s) = \frac{\partial \Phi_a(s)}{\partial a} G_p(s) + b \frac{\partial \Phi_a(s)}{\partial b} \phi_a(s) = \frac{a}{1 + G_c(s)G_p(s)H(s)G_p(s)} \frac{\partial G_p(s)}{\partial a} \\
S_{\phi b}^p(s) &= \frac{\partial \Phi_b(s)}{\partial b} \phi_b(s) = \frac{\partial \Phi_b(s)}{\partial b} G_p(s) + a \frac{\partial \Phi_b(s)}{\partial a} \phi_b(s) = \frac{b}{1 + G_c(s)G_p(s)H(s)G_p(s)} \frac{\partial G_p(s)}{\partial b} \\
S_{\phi a}^c(s) &= \frac{\partial \Phi_a(s)}{\partial a} \phi_a(s) = \frac{\partial \Phi_a(s)}{\partial a} G_C(s) + b \frac{\partial \Phi_a(s)}{\partial b} \phi_a(s) = \frac{a}{1 + G_c(s)G_p(s)H(s)G_p(s)} \frac{\partial G_C(s)}{\partial a} \\
S_{\phi b}^c(s) &= \frac{\partial \Phi_b(s)}{\partial b} \phi_b(s) = \frac{\partial \Phi_b(s)}{\partial b} G_C(s) + a \frac{\partial \Phi_b(s)}{\partial a} \phi_b(s) = \frac{b}{1 + G_c(s)G_p(s)H(s)G_p(s)} \frac{\partial G_C(s)}{\partial b} \\
S_{\phi a}^{Gs}(s) &= \frac{\partial \Phi_a(s)}{\partial a} \phi_a(s) = \frac{\partial \Phi_a(s)}{\partial a} G(s) + b \frac{\partial \Phi_a(s)}{\partial b} \phi_a(s) = \frac{a}{1 + G_c(s)G_p(s)H(s)G_p(s)} \frac{\partial G_p(s)}{\partial a} \\
S_{\phi b}^{Gs}(s) &= \frac{\partial \Phi_b(s)}{\partial b} \phi_b(s) = \frac{\partial \Phi_b(s)}{\partial b} G(s) + a \frac{\partial \Phi_b(s)}{\partial a} \phi_b(s) = \frac{b}{1 + G_c(s)G_p(s)H(s)G_p(s)} \frac{\partial G_p(s)}{\partial b}
\end{align*}
\]

(9)

Once \( G_p(s), G_c(s), \) and \( H(s) \) are given, all above sensitivity functions can be solved. Furthermore, according to (4), all sensitivity functions of frequency characteristics can also be solved.

**Analyzing Frequency Characteristics of Closed-Loop Systems by Sensitivity Function**

Giving a typical closed-loop control system as shown in Figure 1, the plant, the designed controller, and the feedback transfer function, \( G_p(s), G_c(s), \) and \( H(s) \) are respectively given as

\[
G_p(s) = \frac{a}{s^2 + s}, \quad G_c(s) = 1 + \frac{K}{s} + 0.35s, \quad H(s) = 1
\]

(10)

then all sensitivity functions can be solved according to (9) as
\[
\begin{align*}
S^a_D(s) &= \frac{\partial \Phi_D(s)}{\partial a} \Phi_D(s) = \frac{s^2(s+1)}{s^3 + s^2 + a(0.35s^2 + s + K)} \\
S^K_D(s) &= \frac{\partial \Phi_D(s)}{\partial K} \Phi_D(s) = \frac{Ks^2(s+1)}{s^3 + s^2 + a(0.35s^2 + s + K)} \\
S^a_E(s) &= \frac{\partial \Phi_E(s)}{\partial a} \Phi_E(s) = \frac{s^2(s+1)}{s^3 + s^2 + a(0.35s^2 + s + K)} \\
S^K_E(s) &= \frac{\partial \Phi_E(s)}{\partial K} \Phi_E(s) = \frac{aK}{s^3 + s^2 + a(0.35s^2 + s + K)} \\
S^a_R(s) &= \frac{\partial \Phi_R(s)}{\partial a} \Phi_R(s) = \frac{a(0.35s^2 + s + K)}{s^3 + s^2 + a(0.35s^2 + s + K)} \\
S^K_R(s) &= \frac{\partial \Phi_R(s)}{\partial K} \Phi_R(s) = \frac{aK}{s^3 + s^2 + a(0.35s^2 + s + K)} \\
S^a_s(s) &= \frac{\partial \Phi_s(s)}{\partial a} \Phi_s(s) = \frac{s^2(s+1)}{s^3 + s^2 + a(0.35s^2 + s + K)} \\
S^K_s(s) &= \frac{\partial \Phi_s(s)}{\partial K} \Phi_s(s) = \frac{aK}{s^3 + s^2 + a(0.35s^2 + s + K)} \\
S^a_K(s) &= \frac{\partial \Phi_K(s)}{\partial a} \Phi_K(s) = \frac{s^2(s+1)}{s^3 + s^2 + a(0.35s^2 + s + K)} \\
S^K_K(s) &= \frac{\partial \Phi_K(s)}{\partial K} \Phi_K(s) = \frac{aK}{s^3 + s^2 + a(0.35s^2 + s + K)} 
\end{align*}
\]

(11)

It can be seen from (11), for the plant parameter \(a\), sensitivity functions of both closed-loop transfer function \(\Phi_D(s)\) and error transfer function \(\Phi_{DE}(s)\) under disturbance \(D(s)\), \(S^a_D(s)\) and \(S^a_E(s)\), are the same as the sensitivity function of closed-loop transfer function \(\Phi_D(s)\) under reference input \(R(s)\), \(S^a_R(s)\). And for the controller parameter \(K\), sensitivity functions of both closed-loop transfer function \(\Phi_D(s)\) and error transfer function \(\Phi_{DE}(s)\) under disturbance \(D(s)\), \(S^K_D(s)\) and \(S^K_E(s)\), are the same as the sensitivity function of error transfer function \(\Phi_{RE}(s)\) under reference input \(R(s)\), \(S^K_R(s)\). Hence, there are four different sensitivity functions in all eight sensitivity functions about the plant parameter \(a\) and the controller parameter \(K\) to be analyzed hereafter. Figure 2 (a), (b), (c), and (d) show four sensitivity functions of closed-loop system and error signal amplitude-frequency characteristics, \(S^K_R(j\omega)\), \(S^a_D(j\omega)\), \(S^a_E(j\omega)\), and \(S^K_E(j\omega)\), respectively, as either plant parameter \(a=2\) and controller parameter \(K=0.01, 0.05, 0.1\) or controller parameter \(K=0.5\) and plant parameter \(a=1, 5, 10\).
Figure 2. Sensitivity functions of frequency characteristics for: (a) closed-loop transfer function $\Phi_R(s)$ under reference input $R(s)$, $S^\Phi_{R}\omega$, (b) closed-loop transfer function $\Phi_D(s)$ under disturbance $D(s)$, $S^\Phi_{D}\omega$, (c) error transfer function $\Phi_{RE}(s)$ under reference input $R(s)$, $S^\Phi_{RE}\omega$, and (d) error transfer function $\Phi_{DE}(s)$ under disturbance $D(s)$, $S^\Phi_{DE}\omega$.

From Figure 2 there can observe that different plant parameter $a$ or controller parameter $K$ will have different effects on frequency characteristics of closed-loop or error transfer functions. (a) $S^\Phi_{R}\omega$ shows that, the larger the controller parameter $K$ is, the higher the closed-loop system bandwidth is yet the resonance at $\omega=1.1$ radian/second also is. (b) $S^\Phi_{a}\omega$ shows that, the higher the plant parameter $a$ is, the stronger the filtering to low-frequency disturbance is and the weaker the anti-disturbance is. (c) $S^\Phi_{RE}\omega$ shows that, the larger the plant parameter $a$ is, the higher the error signal bandwidth is and the resonance frequency also is. (d) $S^\Phi_{K}\omega$ shows that, the larger the controller parameter $K$ is, the lower the error signal bandwidth is and the stronger the filtering to high-frequency disturbance is, as well as the anti-disturbance also is.

Summary

The sensitivity function is used to analyze the closed-loop frequency characteristic and the error frequency characteristic while the controlled plant parameter or controller parameter are varying. The closed-loop transfer function and the error transfer function are solved under considering the typical closed-loop system which is being acted by the reference input and the disturbance input. Correspondingly, the sensitivity functions of the closed-loop and the error frequency characteristics are derived by partially differentiating the closed-loop transfer function and the error transfer function with respect to the plant parameter and the controller parameter. And then the above sensitivity functions of amplitude-frequency characteristics are compared by analyzing the concrete example in which the plant parameter or controller parameter are given different values. Thus, the specifications of the amplitude-frequency characteristics, such as the closed-loop bandwidth and the resonance frequency, can be observed intuitively. This analysis method can be used to design the plant parameters and determine the controller parameters for closed-loop control systems.

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References


