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An Approximate Analytical Solution to the Problem of Natural Convection of a Newtonian Liquid in a Rectangular Area

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ABSTRACT

The problem of thermoconcentration convection of a viscous incompressible liquid in a rectangular area is solved analytically for a given heat flux through the boundary based on the physical linearization of the hydrodynamic subtask in the Stokes approximation. The solution is presented in form of convergent two-fold series. The computational experiment has confirmed the correctness of the accepted assumptions application.

PROBLEM FORMULATION

The problem of thermoconcentration convection of a viscous incompressible liquid in a rectangular area with a given heat flux through the boundary is considered. At the initial moment of time, the temperature and concentration of the impurity in the volume of the liquid are homogeneous and constitute t_0 and c_0, respectively. The mode of flow is assumed to be laminar. The Novier-Stokes equations are written in the Stokes form [1] under the assumption that $|V_X| << 1$, $|V_Y| << 1$ [2]:

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$$\rho\frac{\partial\bar{v}}{\partial\tau}=-\nabla p+\mu\,\Delta\bar{v}-\rho\beta(t-t_0)\bar{g}+\gamma\rho(c-c_0)\bar{g};\qquad(1)$$

$$\operatorname{div}\bar{v}=0;\qquad(2)$$

$$\frac{\partial t}{\partial\tau}=a\Delta t;\qquad(3)$$

$$\frac{\partial c}{\partial\tau}=D\Delta c,\qquad(4)$$

where □ is the Hamilton operator, Δ is the Laplace operator, $\bar{v}$, p, t, c, are the velocity vector, pressure, temperature, concentration; ρ, β - density and coefficient of volumetric medium expansion; τ - current time; $\bar{g}$ - gravitational vector.

In the component listing form, the system (1)-(4) is presented in the following dimensionless form for the current function, temperature and concentration:

$$\frac{\partial}{\partial\theta}\left(\frac{\partial^2\Psi}{\partial X^2}+\frac{\partial^2\Psi}{\partial Y^2}\right)=\frac{\partial^4\Psi}{\partial X^4}+2\frac{\partial^4\Psi}{\partial X^2\partial Y^2}+\frac{\partial^4\Psi}{\partial Y^4}-\mathrm{Gr}_T\frac{\partial T}{\partial X}+\mathrm{Gr}_C\frac{\partial C}{\partial X};\qquad(5)$$

$$\frac{\partial T}{\partial\theta}=\frac{1}{\mathrm{Pr}}\left(\frac{\partial^2 T}{\partial X^2}+\frac{\partial^2 T}{\partial Y^2}\right);\qquad(6)$$

$$\frac{\partial C}{\partial\theta}=\frac{1}{\mathrm{Sc}}\left(\frac{\partial^2 C}{\partial X^2}+\frac{\partial^2 C}{\partial Y^2}\right)-A^*\exp(A\theta);\qquad(7)$$

$$\Psi(X,Y,0)=0;\qquad(8)$$

$$\Psi(0,Y,\theta)=\Psi(d,Y,\theta)=\Psi(X,0,\theta)=\Psi(X,b,\theta)=0;\qquad(9)$$

$$\frac{\partial\Psi(0,Y,\theta)}{\partial X}=\frac{\partial\Psi(d,Y,\theta)}{\partial X}=\frac{\partial\Psi(X,0,\theta)}{\partial Y}=\frac{\partial^2\Psi(X,b,\theta)}{\partial Y^2}=0;\qquad(10)$$

$$T(X,Y,0)=0;\qquad(11)$$

$$-\frac{\partial T(0,Y,\theta)}{\partial X}=\frac{\partial T(d,Y,\theta)}{\partial X}=-\frac{\partial T(X,0,\theta)}{\partial Y}=1; \qquad (12)$$

$$\frac{\partial T(X,b,\theta)}{\partial Y}=0; \qquad (13)$$

$$C(X,Y,0)=0; \qquad (14)$$

$$C(0,Y,\theta)=C(d,Y,\theta)=C(X,0,\theta)=0; \qquad (15)$$

$$\frac{\partial C(X,b,\theta)}{\partial Y}=0, \qquad (16)$$

where $X=\frac{x}{l}$, $Y=\frac{y}{l}$, $\theta=\frac{\tau\nu}{l^2}$, $\varsigma=\frac{h_1}{h_2}$, $T=\frac{t\lambda}{ql}$, $\mathrm{Gr}_T=\frac{\beta gql^4}{\nu^2\lambda}$, $\mathrm{Gr}_C=\frac{\gamma gql^3}{\nu^2\lambda}$, $l=\frac{2h_1h_2}{h_1+h_2}$, $\mathrm{Pr}=\frac{\nu}{a}$ - Prandtl number; $\mathrm{Sc}=\frac{\nu}{D}$ - Schmidt number; ν, a, D are the kinematic coefficients of viscosity, thermal conductivity and diffusions; Ψ - is the dimensionless current function $V_X=\frac{\partial\Psi}{\partial Y}$, $V_Y=-\frac{\partial\Psi}{\partial X}$; x, y - Cartesian coordinates; h_1, h_2 - width and height of the area; λ, γ are the coefficients of thermal conductivity and kinematic viscosity of the medium, respectively; V_X, V_Y are the projections of the velocity vector on the axes X and Y, q is the heat flux density through the wetted area boundary; A^*, A are the constants characterizing the kinetics of sediment dissolution, $d=\frac{\varsigma+1}{2}$, $b=\frac{\varsigma+1}{2\varsigma}$.

RESULTS

The disconjugate nature of system (5)-(16) allows one to use the solutions of the thermal and concentration problems when finding the dimensionless current function Ψ independently.

Thus, the task can be divided into three subtasks: 1) thermal; 2) concentration; 3) thermoconcentration.

To solve the thermal subtask to equations (6), (11)-(13), we apply the Fourier sine transformation successively according to X [3]:

$$T(X,Y,\theta)=\Lambda(X,Y,\theta)+\frac{2}{\varsigma+1}X^{2}-X+\frac{1}{2}(Y-1)^{2}-\frac{\varsigma-1}{\varsigma+1}(1-Y)\exp(Y-b), \qquad (17)$$

where the function $\Lambda(X,Y,\theta)$ is structurally double Fourier series.

The concentration subtask is represented by equations (7), (14)-(16).

Applying the Fourier sine transform according to *X*, we obtain the solution in the following form:

$$C(X,Y,\theta)=2A^{*}\,\mathrm{Sc}\sum_{k=1}^{\infty}\frac{\cos(\alpha_k d)-1}{\alpha_k}\left\{\frac{1}{\mathrm{A}\cdot\mathrm{Sc}+\alpha_k^{2}}\exp(\mathrm{A}\theta)-\right.$$

$$-\frac{\mathrm{ch}\left[\sqrt{A\cdot\mathrm{Sc}+\alpha_k^{2}}\,(b-Y)\right]}{\left(A\cdot\mathrm{Sc}+\alpha_k^{2}\right)\mathrm{ch}\sqrt{A\cdot\mathrm{Sc}+\alpha_k^{2}}}\exp(\mathrm{A}\theta)-$$

$$\left.-2\sum_{g=1}^{\infty}\frac{\cos\left[\frac{\beta_g}{b}(b-Y)\right]}{\left(\beta_g^{2}+\alpha_k^{2}+A\mathrm{Sc}\cdot b^{2}\right)\beta_g\sin\beta_g}\exp\left(-\frac{\frac{\beta_g^{2}}{b^{2}}+\alpha_k^{2}}{\mathrm{Sc}}\theta\right)\right\}\sin(\alpha_k X). \qquad (18)$$

Let’s consider the third subtask—thermoconcentration.

Applying the final integral sine transform again with respect to *X* and *Y* to equations (5), (8)-(10), we obtain the solution:

$$\Psi(X,Y,\theta)=4\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{p_i^{2}+\eta_j^{2}}\Big[p_i\left(\cos(p_i d)\cdot L_j-Q_j\right)\cdot M\left(p_i,\eta_j,\theta\right)-\eta_j\cdot F_i\cdot N\left(p_i,\eta_j,\theta\right)+$$

$$+\exp\left[-\left(p_i^{2}+\eta_j^{2}\right)\right]\cdot\int_{0}^{\theta}\Phi(\theta)\cdot\exp\left(p_i^{2}+\eta_j^{2}\right)\mathrm{d}\theta\Big]\sin(p_i\,\mathrm{X})\sin(\eta_j\,\mathrm{Y}), \qquad (19)$$

where

$$M\left(p_i,\eta_j,\theta\right)=\frac{1-\exp\left[-\left(p_i^{2}+\eta_j^{2}\right)\theta\right]}{p_i^{2}+\eta_j^{2}}-\frac{\exp\left(-\eta_j^{2}\theta\right)-\exp\left[-\left(p_i^{2}+\eta_j^{2}\right)\theta\right]}{p_i^{2}};$$

$$N\left(p_i,\eta_j,\theta\right)=\frac{1-\exp\left[-\left(p_i^2+\eta_j^2\right)\theta\right]}{p_i^2+\eta_j^2}-\frac{\exp\left(-p_i^2\,\theta\right)-\exp\left[-\left(p_i^2+\eta_j^2\right)\theta\right]}{\eta_j^2};$$

$$\sin\left(p_i d\right)=0;\ \sin\left(\eta_j b\right)=0.$$

The coefficients L_j, Q_j and F_i in expression (19) are found from the system of equations (20), which is obtained after satisfying (19) the boundary conditions (9)-(10):

$$\sum_{i=1}^{\infty}\frac{p_i}{p_i^2+\eta_j^2}\left\{\left[p_l\cos\left(p_i d\right)\cdot L_j-Q_j\right]\cdot M\left(\xi_l,\eta_j,\theta\right)-\eta_j\cdot F_i\cdot N\left(p_l,\eta_j,\theta\right)+\right.$$

$$\left.+\exp\left[-\left(p_i^2+\eta_j^2\right)\theta\right]\cdot\int_0^{\theta}\Phi(\theta)\cdot\exp\left[\left(p_i^2+\eta_j^2\right)\theta\right]\mathrm{d}\theta\right\}=0;$$

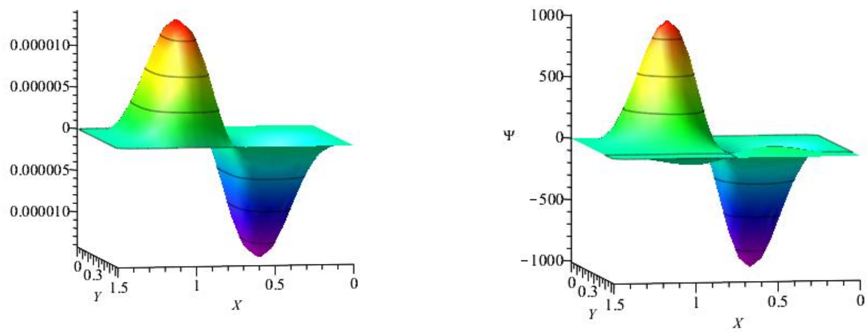
$$\sum_{i=1}^{\infty}\frac{p_i\cos\left(p_i d\right)}{p_i^2+\eta_j^2}\left\{\left[p_l\cos\left(p_i d\right)\cdot L_j-Q_j\right]\cdot M\left(\xi_l,\eta_j,\theta\right)-\eta_j\cdot F_i\cdot N\left(p_l,\eta_j,\theta\right)+\right.$$

$$\left.+\exp\left[-\left(p_i^2+\eta_j^2\right)\theta\right]\cdot\int_0^{\theta}\Phi(\theta)\cdot\exp\left[\left(p_i^2+\eta_j^2\right)\theta\right]\mathrm{d}\theta\right\}=0; \qquad (20)$$

$$\sum_{i=1}^{\infty}\frac{\eta_j}{p_i^2+\eta_j^2}\left\{\left[p_l\cos\left(p_i d\right)\cdot L_j-Q_j\right]\cdot M\left(\xi_l,\eta_j,\theta\right)-\eta_j\cdot F_i\cdot N\left(p_l,\eta_j,\theta\right)+\right.$$

$$\left.+\exp\left[-\left(p_i^2+\eta_j^2\right)\theta\right]\cdot\int_0^{\theta}\Phi(\theta)\cdot\exp\left[\left(p_i^2+\eta_j^2\right)\theta\right]\mathrm{d}\theta\right\}=0. \qquad (21)$$

CONCLUSIONS



a b

Figure 1. Dynamics of the free-convective flow development in a rectangular area: at $\mathrm{Pr} = 1{,}01$, $\mathrm{Sc} = 2{,}1$, $\mathrm{Gr}_T = 1$, $\mathrm{Gr}_C = 4{,}84 \cdot 10^{-5}$ for different θ: a) – 0,002; b) – 0,05.

The result of the computational experiment is presented in Figure 1, which shows the dynamics corresponding to the well-known ideas about the free-convective flow [4].

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