Periodic Solution of a Nonautonomous Predator-Prey System of Holling Type II with Strong Allee Effect and Impulsive Perturbations

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Abstract. The principle of this paper is to explore the existence of periodic solution of a nonautonomous predator-prey system of Holling type II with strong Allee effect and impulsive perturbations. Sufficient and realistic conditions are obtained by using Mawhin’s continuation theorem of coincidence degree and analysis techniques. Further, some numerical simulations demonstrate our results.

Introduction

The predator-prey system is one of the most important topics in biological mathematics and has been widely discussed by many scholars [1-4]. In real situation, biological populations may exhibit Allee effect [5], which means the populations decrease in size if the population falls below a certain threshold. In literature [6, 7], the predator-prey system with a strong Allee effect are studied. However, many systems arising in physical, biological phenomena exhibit impulsive dynamics behaviors due to the abrupt jumps during the evolution processes, which can be modelled by impulsive differential equations [8]. In this paper, we shall explore the dynamic of a nonautonomous predator-prey system of Holling type II with strong Allee effect and impulsive perturbation:

\[
\begin{align*}
\begin{cases}
x_1'(t) &= r(t)x_1(t)(x_1(t) - \beta(t))(1 - \frac{x_1(t)}{k(t)}) - \frac{c(t)x_1(t)x_2(t)}{a(t) + x_1(t)},
\quad t \neq t_k, \\
x_2'(t) &= x_2(t)\left(\frac{\alpha(t)x_1(t)}{a(t) + x_1(t)} - d(t)\right), \\
\Delta x_1(t_k) &= x_1(t_k^+) - x_1(t_k^-) = h_1^k x_1(t_k), \\
\Delta x_2(t_k) &= x_2(t_k^+) - x_2(t_k^-) = h_2^k x_2(t_k),
\end{cases}
\end{align*}
\]

where \( x_1(t) \) and \( x_2(t) \) represent densities of prey and predator at time \( t \) respectively; \( r(t), \beta(t), k(t), \alpha(t), c(t), d(t) \) are all positive \( \omega \)-periodic functions, represent the intrinsic growth rate of the prey, the threshold value of Allee effect, the environment capacity of the prey, conversion rate, assimilation rate and the mortality of the predator respectively. \( h_1^k, h_2^k \) are constants; when \( h_1^k < 0 \), it represents the harvest rate, otherwise represents the delivery rate at time \( t \). \( 1 + h_2^k > 0 \) and there exists a positive integer \( q \) such that \( t_{k+q} = t_k + \omega \), \( h_1^{t+q} = h_1^k \), \( h_2^{t+q} = h_2^k \).

Existence of Positive Periodic Solutions

First we shall make some preparations. Let \( J \subset \mathbb{R} \), denote by \( PC(J, \mathbb{R}^N) \) the space of functions \( \phi : J \to \mathbb{R}^N \) which are continuous at \( t \in J \), \( t \neq t_k \) and left continuous at \( t = t_k \).
Let \( g(t) \) be an \( \omega \)-periodic function. For the sake of convenience, we set
\[
g' = \inf_{t \in [0,a]} \{ g(t) \}, \quad g'' = \sup_{t \in [0,a]} \{ g(t) \}, \quad \overline{g} = \frac{1}{\omega} \int_0^\omega g(t)dt.
\]

To prove our results, we also need a few concepts from Gaines and Mawhin [9,10].

**Lemma 1.** ([9]) Let \( X \) and \( Y \) be two Banach space. Consider an operator equation \( Lx = \lambda Nx \), where \( L: \text{Dom} L \cap X \rightarrow Y \) is a Fredholm operator of index zero and \( \lambda \in [0,1] \) is a parameter. Then there exist two projectors \( P : X \rightarrow X \) and \( Q : Y \rightarrow Y \) such that \( \text{Im} P = \ker L \) and \( \text{Im} L = \ker Q \).

Suppose \( N : \overline{\Omega} \rightarrow Y \) is \( L \)-compact on \( \overline{\Omega} \), where \( \Omega \) is open bounded in \( X \). Furthermore, assume:

(a) for each \( \lambda \in (0,1) \), \( x \in \partial \Omega \cap \text{Dom} L \), \( Lx \neq \lambda Nx \); 
(b) for each \( x \in \partial \Omega \cap \ker L \), \( QNx \neq 0 \);
(c) \( \deg \{ JQN, \Omega \cap \ker L, 0 \} \neq 0 \), where \( J : \text{Im} Q \rightarrow \ker L \) is an isomorphism and \( \deg \{ \ast \} \) represents the Brouwer degree.

Then equation \( Lx = Nx \) has at least one solution on \( \partial \Omega \cap \text{Dom} L \).

Now we are ready to state and prove our main result.

**Theorem 1.** Suppose system (1) satisfies the following conditions:

\[
\begin{align*}
& (H_1) \quad \overline{d} \omega - \sum_{k=1}^q \ln(1 + h_k^2) > 0; \quad (H_2) \quad \overline{a} \omega - \overline{d} \omega + \sum_{k=1}^q \ln(1 + h_k^2) > 0; \\
& (H_3) \quad \overline{r} \omega \left( \exp \{ E_1 \} - \beta^u \right) 1 - \frac{\exp \{ D_1 \}}{k} \right) + \sum_{k=1}^q \ln(1 + h_k^2) > 0,
\end{align*}
\]

where \( D_1 = B_1 + \frac{1}{2} A_1 + \frac{1}{2} \sum_{k=1}^q \ln(1 + h_k^1) \), \( E_1 = C_1 - \frac{1}{2} A_1 = \frac{1}{2} \sum_{k=1}^q \ln(1 + h_k^1) \), 
\[
A_1 = 2 \overline{r} \beta^u \omega + \sum_{k=1}^q \ln(1 + h_k^1) \right) - \sum_{k=1}^q \ln(1 + h_k^1),
\]

\[
\begin{align*}
& C_1 = \ln \left( \overline{a} \omega - \overline{d} \omega + \sum_{k=1}^q \ln(1 + h_k^2) \right), \\
& B_1 = \ln \left( \overline{a} \omega - \overline{d} \omega + \sum_{k=1}^q \ln(1 + h_k^2) \right), \\
& f(t) = \frac{\alpha(t)}{\beta(t)}.
\end{align*}
\]

Then system (1) has at least one positive \( \omega \)-periodic solution.

**Proof.** Let \( x_i(t) = \exp \{ y_i(t) \} \), \( x_2(t) = \exp \{ y_2(t) \} \), then system (1) can be reformulated as
\[
\begin{align*}
\left\{ \begin{array}{l}
y_1'(t) = r(t) \left( \exp \{ y_1(t) \} - \beta(t) \right) \left( 1 - \frac{\exp \{ y_1(t) \}}{k(t)} \right) - \frac{c(t) \exp \{ y_2(t) \}}{a(t) + \exp \{ y_1(t) \}} \right) : = f_1(t), \\
y_2'(t) = \frac{\alpha(t) \exp \{ y_1(t) \}}{a(t) + \exp \{ y_1(t) \}} - d(t) : = f_2(t), \\
\Delta y_1(t_k) = y_1(t_k^+) - y_1(t_k^-) = \ln(1 + h_k^1), \\
\Delta y_2(t_k) = y_2(t_k^+) - y_2(t_k^-) = \ln(1 + h_k^2), \\
\end{array} \right. \\
t \neq t_k,
\end{align*}
\]

Define \( X = \{ x(t) \in PC(R, R^2) : x(t + \omega) = x(t) \} \), \( Y = \{ y = (\tilde{y}, \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_q) \} = X \times R^{2q} \), then it is standard to show that both \( X \) and \( Y \) are Banach space when they are endowed with the norm
\[
\| x \| = \| x_1(t) \|_1 + \max_{t \in [0,a]} \| x_2(t) \|_1,
\]

\[
\| \tilde{y} \| = \max_{t \in [0,a]} \| \tilde{y}(t) \| + \sum_{i=1}^q \max_{t \in [0,a]} \| \tilde{y}_i(t) \|.
\]

Set \( L: \text{Dom} L \subset X \rightarrow Y \) and \( N: X \rightarrow Y \) as
\[
L \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y'_1(t) \\ y'_2(t) \end{pmatrix} \begin{pmatrix} \Delta y_1(t_1) \\ \Delta y_2(t_1) \\ \vdots \\ \Delta y_1(t_q) \\ \Delta y_2(t_q) \end{pmatrix}, \quad N(y(t)) = \begin{pmatrix} f_1 \\ f_2 \\ \ln(1+h^1_k) \\ \vdots \\ \ln(1+h^q_k) \end{pmatrix}.
\]

It is easily to prove that \( L \) is a Fredholm operator of index zero and \( N \) is \( L \)-compact on \( \Omega \) for any given open and bound subset \( \Omega \) in \( X \). Consider the operator equation

\[
Ly = \lambda Ny, \quad \lambda \in (0,1).
\]

Suppose \( y(t) = (y_1(t), y_2(t))^T \) is a periodic solution of (3) for certain \( \lambda \in (0,1) \). By integrating (3) over the interval \([0, \omega]\), we obtain

\[
\int_0^\omega \frac{c(t) \exp\{y_2(t)\}}{a(t) + \exp\{y_1(t)\}} dt = \int_0^\omega \left[ r(t) \left( \exp\{y_1(t)\} - \beta(t) \right) \left( 1 - \frac{\exp\{y_1(t)\}}{k(t)} \right) \right] dt + \sum_{k=1}^q \ln(1+h^1_k) \tag{4}
\]

\[
\int_0^\omega \frac{\alpha(t) \exp\{y_1(t)\}}{a(t) + \exp\{y_1(t)\}} dt = \tilde{d} \omega - \sum_{k=1}^q \ln(1+h^2_k). \tag{5}
\]

From (3), (4) and (5), we can derive

\[
\int_0^\omega |y'_1(t)| dt \leq 2\tilde{\beta}^* \omega + \sum_{k=1}^q \left| \ln(1+h^1_k) \right| - \sum_{k=1}^q \ln(1+h^1_k) := A_1, \tag{6}
\]

\[
\int_0^\omega |y'_2(t)| dt \leq 2\tilde{\omega} + \sum_{k=1}^q \left| \ln(1+h^2_k) \right| - \sum_{k=1}^q \ln(1+h^2_k) := A_2. \tag{7}
\]

Since \( y_i(t) \in PC([0, \omega], R) \), there exist \( \xi_i, \eta_i \in [0, \omega] \), such that

\[
y_i(\xi_i) = \inf_{t \in [0, \omega]} y_i(t), \quad y_i(\eta_i) = \sup_{t \in [0, \omega]} y_i(t), \quad i = 1, 2.
\]

From (5), we can see

\[
\int_0^\omega \frac{\alpha(t) \exp\{y_1(\xi_i)\}}{a(t) + \exp\{y_1(\xi_i)\}} dt \leq \int_0^\omega \frac{\alpha(t) \exp\{y_1(t)\}}{a(t) + \exp\{y_1(t)\}} dt = \tilde{d} \omega - \sum_{k=1}^q \ln(1+h^2_k),
\]

\[
\int_0^\omega \frac{\alpha(t) \exp\{y_1(\eta_i)\}}{a(t) + \exp\{y_1(\eta_i)\}} dt \geq \int_0^\omega \frac{\alpha(t) \exp\{y_1(t)\}}{a(t) + \exp\{y_1(t)\}} dt = \tilde{d} \omega - \sum_{k=1}^q \ln(1+h^2_k),
\]

Define \( f(t) = \frac{\alpha(t)}{a(t)} \), it follows that

\[
y_1(\xi_i) \leq \ln \frac{\tilde{d} \omega - \sum_{k=1}^q \ln(1+h^2_k)}{\tilde{\alpha} \omega - \tilde{d} \omega + \sum_{k=1}^q \ln(1+h^2_k)} := B_i, \quad y_1(\eta_i) \geq \ln \frac{\tilde{d} \omega - \sum_{k=1}^q \ln(1+h^2_k)}{f(\omega) \tilde{d} \omega - \tilde{d} \omega + \sum_{k=1}^q \ln(1+h^2_k)} := C_i.
\]

Then for \( \forall t \in [0, \omega] \), by lemma 2.2 of [10], we have

\[
y_1(t) \leq y_1(\xi_i) + \frac{1}{2} \left[ \int_0^\omega |y'_1(t)| dt + \sum_{k=1}^q \left| \ln(1+h^1_k) \right| \right] \leq B_1 + \frac{1}{2} A_1 + \frac{1}{2} \sum_{k=1}^q \left| \ln(1+h^1_k) \right| := D_1
\]

\[
y_1(t) \geq y_1(\eta_i) - \frac{1}{2} \left[ \int_0^\omega |y'_1(t)| dt + \sum_{k=1}^q \left| \ln(1+h^1_k) \right| \right] \geq C_1 - \frac{1}{2} A_1 - \frac{1}{2} \sum_{k=1}^q \left| \ln(1+h^1_k) \right| := E_1
\]

Similarly, from (4), we get
It follows that
\[
\begin{aligned}
\int_0^\omega c(t) \exp\{y_2(t)\} \, dt &\leq \int_0^\omega c(t) \exp\{y_1(t)\} \, dt \\
&\leq \int_0^\omega r(t) \exp\{y_1(t)\} + r(t) \beta(t) \frac{\exp\{y_1(t)\}}{k(t)} \, dt + \sum_{k=1}^q \ln(1 + h_k^1) \\
&\leq \exp\{D_1\} r \omega + \frac{\exp\{D_1\}}{k^1} r \beta \omega + \sum_{k=1}^q \ln(1 + h_k^1),
\end{aligned}
\]
\[
\int_0^\omega c(t) \exp\{y_2(t)\} \, dt \geq \int_0^\omega c(t) \exp\{y_1(t)\} \, dt \\
&\geq \overline{r} \omega \{ \exp\{E_1\} - \beta^* \} \left(1 - \frac{\exp\{D_1\}}{k^1}\right) + \sum_{k=1}^q \ln(1 + h_k^1).
\]

It follows that
\[
\begin{aligned}
y_2(\xi_2^1) &\leq \ln \frac{\exp\{D_1\} \overline{r} \omega + \frac{\exp\{D_1\}}{k^1} r \beta \omega + \sum_{k=1}^q \ln(1 + h_k^1)}{\overline{r} \omega} := B_2, \\
y_2(\eta_2) &\geq \ln \frac{\exp\{E_1\} \overline{r} \omega \{ \exp\{E_1\} - \beta^* \} \left(1 - \frac{\exp\{D_1\}}{k^1}\right) + \sum_{k=1}^q \ln(1 + h_k^1)}{\overline{r} \omega} := C_2.
\end{aligned}
\]

So, by lemma 2.2 of [10], we have
\[
\begin{aligned}
y_2(t) &\leq y_2(\xi_2^1) + \frac{1}{2} \int_0^\omega \dot{y}_2(t) \, dt + \sum_{k=1}^q \ln(1 + h_k^2) \mid \equiv D_2 \\
y_2(t) &\geq y_2(\eta_2) - \frac{1}{2} \int_0^\omega \dot{y}_2(t) \, dt + \sum_{k=1}^q \ln(1 + h_k^2) \mid \equiv E_2.
\end{aligned}
\]

Thus we can derive
\[
\mid y_1(t) \mid \leq \max\{\mid D_1 \mid, \mid E_1 \mid\} := G_1, \quad \mid y_2(t) \mid \leq \max\{\mid D_2 \mid, \mid E_2 \mid\} := G_2.
\]

Obviously, \(G_1, G_2\) are independent of \(\lambda\). Therefore, there exists a constant \(G_3 > 0\), satisfying \(\max\{\mid y_1 \mid, \mid y_2 \mid\} \leq G_3\). Let \(r > G_1 + G_2 + G_3\), \(\Omega = \{y \in X : \mid y \mid < r\}\), then \(\Omega\) satisfies condition (a) of lemma 1. When \(y = (y_1, y_2)^T \in \partial \Omega \cap \ker L = \partial \Omega \cap R^2\), \(y\) is the constant vector in \(R^2\) with \(\mid y \mid = r\), then it is easy to prove that \(QNx \neq 0\). In addition, a direct computation gives \(\deg(\text{JQN} |_{\partial \Omega \cap \ker L}) \neq 0\). By now, \(\Omega\) satisfies all the requirements in Mawhin’s continuation theorem (Lemma 1), hence model (2) has at least one \(\omega\)-periodic solution \(y^*(t) \in (y_1^*(t), y_2^*(t))^T \in \overline{\Omega} \cap \text{Dom} L\). Set \(x_1^*(t) = \exp\{y_1^*(t)\}\), \(x_2^*(t) = \exp\{y_2^*(t)\}\), then \(x^*(t) \in (x_1^*(t), x_2^*(t))^T\) is an positive \(\omega\)-periodic solution of system (1). The proof is complete.

**An Illustrative Example**

In system (1), we take \(t_k = kT\), \(r(t) = 0.5 + 0.1 \cos t\), \(\beta(t) = 0.2 + 0.05 \cos t\), \(k(t) = 2 + 0.1 \cos t\), \(c(t) = 0.3 + 0.1 \sin t\), \(a(t) = 1 + 0.1 \cos t\), \(\alpha(t) = 0.2 + 0.05 \sin t\), \(d(t) = 0.1 + 0.05 \cos t\), \(h_k^1 = -0.2\) and \(h_k^2 = -0.05\).

If \(T = \pi\), then all conditions of Theorem 1 are satisfied, system (1) has a \(2\pi\)-periodic solution (see Fig. 1-3, we take \(x_1(0) = 0.8\), \(x_2(0) = 0.7\)). The influence of the period pulses is obvious.
If $T = 2$, then $h_{k+q}^1 = h_k^1, h_{k+q}^2 = h_k^2$ are not satisfied. Periodic oscillation of system (1) will be destroyed by impulsive effect. Numeric results show that system (1) has a Gui chaotic strange attractor as in [8] (see Fig. 4, we take $x_1(0) = 1.1, x_2(0) = 1$).

Figure 1. Time-series of $x_1(t)$ evolved in system (1).
Figure 2. Time-series of $x_2(t)$ evolved in system (1).
Figure 3. Phase portrait of periodic solution of system (1).
Figure 4. Phase portrait of strange attractor of system (1).

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References


