A Priori Indicator of Convergence Rate for Power Method for Computing the Spectral Radius of Saddle Point Matrices

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ABSTRACT

Saddle point matrix is the coefficient matrix of the linear system derived from the saddle point problem, which arise from many scientific and engineering applications. The eigenvalue information of the saddle point matrices is usually quite important in practical computation. In this paper, the power method is applied to computing the spectral radius of the saddle point matrices. Furthermore, based on some theoretical results of eigenvalue estimates for the saddle point matrices, a new indicator, which is a function of the extreme eigenvalues of the sub blocks, is proposed to predict the convergence rate of the power method. The numerical results of the experiment on computing the spectral radius of the saddle point matrices derived from the model of the Stokes equation demonstrate that the proposed algorithm and the indicator are both effective.

1. INTRODUCTION

Saddle point problems arise from many scientific research fields and engineering applications, such as mixed finite element methods, constrained least square problems, image processing and so on [1-16], and usually generate the following linear system:

\[
\begin{pmatrix}
A & B \\
B^T & O
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
b \\
q
\end{pmatrix}.
\]

(1)

This system is also called “augmented system” [1, 4] or “KKT system” [1], and its coefficient matrix

\[
W = \begin{pmatrix}
A \\
B^T \\
O
\end{pmatrix}
\]

(2)

is called “saddle point matrix” [10-16]. Here the sub block \( A \in \mathbb{R}^{m \times m} \) is symmetric and positive definite, \( B \in \mathbb{R}^{m \times n} \) \((m \geq n)\) has full column rank, and \( O \) is an n-order
zero matrix. Matrix $W$ is symmetric and indefinite [1], and has a peculiar block structure. In order to make better use of the sparsity of $W$ in large-scale computation, researchers have developed various efficient algorithms over the past 30 years [1-8]. As far as we know, the focus of current research has shifted to the preconditioning techniques for accelerating the convergence rate of the iteration algorithms. Most of these techniques need eigenvalue information of matrix $W$ (or other related matrices). Therefore, the research on the properties of the key eigenvalues of saddle point matrix has attracted more attentions [9-16]. We should note that in practice the size of matrices are usually very large, and the computational complexity should be as small as possible. On the other hand, sometimes the eigenvalues of sub blocks $A$ and $B$ may be obtained in advance by some simple methods. The inexpensive information should be fully utilized.

In this paper, we introduce the power method to compute the spectral radius, i.e., the maximum eigenvalue modulus of the saddle point matrix $W$, and establish a concrete program according to the specific structure of $W$. Furthermore, based on the theoretical results of some classical references we propose a new priori indicator, which depends on the maximum and minimum eigenvalues (or singular values) of sub blocks $A$ and $B$, to predict the convergence rate of power method. In the experiment, we test this method with P1-P0 mixed finite element model. Numerical results demonstrate that the involved algorithm and indicator are both valid.

We use following notations throughout this article. The eigenvalues of saddle point matrix $W$ are denoted in descending order as: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m+n}$. The symbols $|\tilde{\lambda}_1|$ and $|\tilde{\lambda}_2|$ represent the largest and the second largest eigenvalue modulus of $W$ respectively. The eigenvalues of the sub block $A$ are denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m (> 0)$, and the singular values of sub block $B$ are denoted by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n (> 0)$. In this article we assume the eigenvalues of $A$ and singular values of $B$ are easily obtained in advance. The notation $|\cdot|$ represents the modulus (absolute value) function of the real number. The sign $\lambda_k(\cdot)$ denotes the k-th largest eigenvalue of the involved (symmetric) matrix. In addition, the notation “$X \preceq Y$” means $Y - X$ is symmetric and semi positive definite.

2. MAIN RESULTS

2.1 Basic Theory on the Eigenvalues of Saddle Point Matrix

We start the discussion with a brief review of some basic theory. First of all, let us recall some classical useful theoretical results for the general symmetric matrices.

Lemma 1 (Corollary of Weyl Theorem) [17] If $X$ and $Y$ are both symmetric matrices, and $X \preceq Y$, then $\lambda_k(X) \leq \lambda_k(Y)$.

Lemma 2 (Sturm) [17] Assume $\overline{W} \in R^{M \times M}$ is symmetric, and $\overline{A}_m$ is an arbitrary $m$-order principle sub matrix of $\overline{W}$, then it holds that

$$\lambda_{M-m+k} (\overline{W}) \leq \lambda_k (\overline{A}_m) \leq \lambda_k (\overline{W}), \quad 1 \leq k \leq m \leq M.$$  

Moreover, for the eigenvalue distribution of saddle point matrix, there is a particularly important result as follows.

Lemma 3 (Rusten)[1, 2] Let saddle point matrix $W$ and its sub blocks $A$ and $B$ defined as (2), and the spectrum of $W$ is denoted by $\Lambda(W)$, then
\[ \Lambda(W) \subseteq I = I^- \cup I^+, \text{ where} \]
\[ I^- = \left\{ \frac{1}{2} \left( \mu_m - \sqrt{\mu_m^2 + 4\sigma_1^2} \right), \frac{1}{2} \left( \mu_1 - \sqrt{\mu_1^2 + 4\sigma_1^2} \right) \right\}, \text{ and} \]
\[ I^+ = \left\{ \mu_m, \frac{1}{2} \left( \mu_1 + \sqrt{\mu_1^2 + 4\sigma_1^2} \right) \right\}. \]

**Remark:** Lemma 2 and Lemma 3 indicate that the eigenvalues of \( W \) and \( A \) satisfy the following relation:
\[ \frac{1}{2} \left( \mu_1 + \sqrt{\mu_1^2 + 4\sigma_1^2} \right) \geq \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \ldots \geq \lambda_m \geq \mu_m > 0 \]
\[ > \frac{1}{2} \left( \mu_1 - \sqrt{\mu_1^2 + 4\sigma_1^2} \right) \geq \lambda_{m+1} \geq \ldots \geq \lambda_{m+n} \geq \frac{1}{2} \left( \mu_m - \sqrt{\mu_m^2 + 4\sigma_1^2} \right). \]

### 2.2 Power Method for Saddle Point Matrix

Power method has a simple scheme and is suitable to compute the largest eigenvalue modulus of the symmetric matrix [18, 19]. According to the structure of saddle point matrix \( W \), we establish a program of power method [18] as follows.

**Algorithm 1. Power Method for Saddle Point Matrix**

1. Input the sub blocks \( A, B \) of matrix \( W \), start with vector \( x = x_0 \in R^m \) and \( y = y_0 \in R^n \); set threshold \( \varepsilon \) for the stop criterion
2. For \( k = 1, 2, \ldots \)
3. \[ u = x/\sqrt{x^T x + y^T y}, \quad v = y/\sqrt{x^T x + y^T y} \]
4. \[ x = Au + Bv, \quad y = B^T u \]
5. \[ \theta = u^T x + v^T y \]
6. If \( \sqrt{(x - \theta u)^T (x - \theta u) + (y - \theta v)^T (y - \theta v)} < \varepsilon |\theta| \), Stop
7. End for
8. Accept \( \hat{\rho} = \theta \) as the approximate value of the largest eigenvalue by modulus of \( W \)

Some upper bound of the convergence rate of the power method is characterized by the ratio \( |\lambda_2|/|\lambda_1| \) [19]. Generally speaking, the smaller the ratio \( |\lambda_2|/|\lambda_1| \) is, the faster the algorithm converges. Unfortunately, the ratio \( |\lambda_2|/|\lambda_1| \) is usually unknown before computation, thus we cannot set it as a priori indicator. Our desire is finding a practical priori indicator to predict the convergence rate of the power method.

### 2.3 A New Indicator of Convergence Rate

Rusten [2] has pointed out that some bounds of Lemma 3 are sharp, that is,
\[ \lambda_1 \approx \frac{1}{2} \left( \mu_1 + \sqrt{\mu_1^2 + 4\sigma_1^2} \right), \quad \lambda_{m+n} \approx \frac{1}{2} \left( \mu_m - \sqrt{\mu_m^2 + 4\sigma_1^2} \right). \]

A large number of numerical results (of this article, including reported and unreported numerical results) support this view. Based on condition (4) we propose following result.

**Proposition 1** Under the assumption of (4), the largest eigenvalue modulus of saddle point matrix \( W \) is \( |\lambda_1| \), and the second largest eigenvalue modulus should be \( \max\{|\lambda_2|, |\lambda_{m+n}|\} \).

**Proof.** Comparing the absolute value of the eigenvalues \( \lambda_1 \) and \( \lambda_{m+n} \) under the assumption of (4), it yields that
\[ |\lambda_1| - |\lambda_{m+n}| \approx \frac{1}{2} \left( \mu_1 + \sqrt{\mu_1^2 + 4\sigma_1^2} \right) - \frac{1}{2} \left( \mu_m - \sqrt{\mu_m^2 + 4\sigma_1^2} \right) \]
\[ = \frac{1}{2} \left( \mu_1 + \sqrt{\mu_1^2 + 4\sigma_1^2} \right) - \frac{1}{2} \left( \mu_m + \sqrt{\mu_m^2 + 4\sigma_1^2} \right) \]
\[ = \frac{1}{2} \left( \mu_1 + \mu_m \right) + \frac{1}{2} \left( \sqrt{\mu_1^2 + 4\sigma_1^2} - \sqrt{\mu_m^2 + 4\sigma_1^2} \right) > 0. \quad (5) \]

which means \( \lambda_1 \) has the largest modulus among all the eigenvalues of \( W \), and then the second largest eigenvalue modulus should be the maximum of \( |\lambda_2| \) and \( |\lambda_{m+n}| \).

The proof is completed.

It follows from Proposition 1 and (4) that
\[ \frac{|\lambda_{m+n}|}{|\lambda_1|} \approx \frac{\frac{1}{2} \left( \sqrt{\mu_m^2 + 4\sigma_1^2} - \mu_m \right)}{\mu_1 + \mu_m + \sqrt{\mu_1^2 + 4\sigma_1^2}} = \frac{n}{\mu_1 + \mu_m + \sqrt{\mu_1^2 + 4\sigma_1^2}}. \quad (5) \]

In addition, (3) shows that \( \lambda_2 \in [\mu_2, \mu_1] \). We use the midpoint of interval \([\mu_2, \mu_1]\) as a probabilistic approximation of the value of \( \lambda_2 \), i.e., \( \lambda_2 \approx \frac{1}{2} (\mu_1 + \mu_2) \). Therefore,
\[ \frac{|\lambda_2|}{|\lambda_1|} \approx \frac{\frac{1}{2} (\mu_1 + \mu_2)}{\mu_1 + \mu_m + \sqrt{\mu_1^2 + 4\sigma_1^2}} = \frac{\mu_1 + \mu_2}{\mu_1 + \mu_m + \sqrt{\mu_1^2 + 4\sigma_1^2}}. \quad (6) \]

According to (5) and (6), now we propose the following indicator:
\[ \chi = \max \left\{ \sqrt{\mu_m^2 + 4\sigma_1^2} - \mu_m, \frac{\mu_1 + \mu_2}{\mu_1 + \mu_m + \sqrt{\mu_1^2 + 4\sigma_1^2}} \right\}. \quad (7) \]

which is expected to be an effective approximation of the ratio \( |\tilde{\lambda}_2|/|\tilde{\lambda}_1| \) since \( \chi \approx \max(|\lambda_{m+n}|/|\lambda_1|) \). The advantage of this new indicator is that it only relies on some eigenvalues or singular values of sub blocks \( A \) and \( B \), and avoids the computation for the eigenvalues of \( W \).

3. NUMERICAL EXPERIMENT

Following stationary Stokes equation is a classical problem in computational fluid dynamics [1, 3, 20]:
\[
\begin{align*}
-\tau \Delta u + \nabla p &= f, \quad \text{in } \Omega; \\
\text{div } u &= 0, \quad \text{in } \Omega; \\
\int_{\Omega} p d\Omega &= 0; \\
|u|_{\partial \Omega} &= g,
\end{align*}
\]
where \( \Omega = (0,1) \times (0,1) \) is a unit square domain, and \( \partial \Omega \) is the boundary of \( \Omega \). Vector \( u \) represents the velocity, and \( p \) stands for the pressure. The constant \( \tau > 0 \) is the viscosity coefficient [1]. We take P1-P0 mixed finite element method on the mixed triangular grids to discrete the stationary Stokes equation and derive the saddle point system as (1) (see [20] for details), which has the coefficient matrix as (2).

We implement several groups of numerical experiments on different levels of grids and viscosity coefficients, and apply Algorithm 1 to computing the spectral radius (largest eigenvalue modulus) of the corresponding saddle point matrices. In each case, Algorithm 1 runs with the initial vectors \( x_0 = (1,1,\ldots,1)^T \), \( y_0 = (1,1,\ldots,1)^T \) and the
threshold $\epsilon = 0.001$. The maximum and minimum eigenvalues of $A$, the maximum and minimum singular values of $B$, the values of ratio $|\tilde{\lambda}_2|/|\tilde{\lambda}_1|$, the values of indicator $\chi$, the maximum eigenvalue of $W$, the computed spectral radius $\hat{\rho}$, the iteration times (IT) and the CPU time cost in each case are all reported in Table 1.

Some interesting phenomena emerged. In comparison, the algorithm converges fastest when $\tau = 0.1$, and slowest when $\tau = 0.001$. The values of the ratio $|\tilde{\lambda}_2|/|\tilde{\lambda}_1|$ and the indicator $\chi$ are very close in each case, and in most case they are monotonically related to the convergence rates. A seemingly “anomalous” phenomenon is that the performance of the ratio $|\tilde{\lambda}_2|/|\tilde{\lambda}_1|$ (and the indicator $\chi$) is not consistent with the convergence rate in case $\tau = 1$. We think the reason is that the ratio $|\tilde{\lambda}_2|/|\tilde{\lambda}_1|$ only determines some upper bound of the convergence rate, so it is possible that the ratio $|\tilde{\lambda}_2|/|\tilde{\lambda}_1|$ has a large value but the convergence rate is still fast. In conclusion, numerical results show that Algorithm 1 is competent for the computation of spectral radius of saddle point matrix, and indicator $\chi$ is an effective substitute for the ratio $|\tilde{\lambda}_2|/|\tilde{\lambda}_1|$.

| $\tau$ | Grid | $\mu_1$ | $\mu_m$ | $\sigma_1$ | $\sigma_n$ | $|\tilde{\lambda}_2|/|\tilde{\lambda}_1|$ | $\chi$ | $\lambda_1$ | $\hat{\rho}$ | IT | CPU time |
|-------|------|--------|--------|----------|----------|----------------|------|----------|------|-----|----------|
| 0.001 | 8 x 8 | 7.695  | 0.304  | 1.031   | 0.079   | 0.998         | 0.982| 987      | 994  | 267| 7.703    |
|       | 16 x 16| 7.923  | 0.076  | 1.007   | 0.020   | 0.999         | 0.984| 771      | 755  | 324| 7.925    |
|       | 32 x 32| 7.980  | 0.019  | 1.001   | 0.005   | 0.999         | 0.984| 954      | 939  | 716| 7.981    |
| 0.01  | 8 x 8  | 0.769  | 0.030  | 1.031   | 0.079   | 0.673         | 0.684| 360      | 679  | 078| 1.268    |
|       | 16 x 16| 0.792  | 0.007  | 1.007   | 0.020   | 0.670         | 0.678| 820      | 771  | 404| 1.243    |
|       | 32 x 32| 0.798  | 0.001  | 1.001   | 0.005   | 0.669         | 0.677| 954      | 282  | 587| 1.236    |
| 0.1   | 8 x 8  | 0.076  | 0.003  | 1.031   | 0.079   | 0.961         | 0.961| 954      | 932  | 392| 1.051    |
|       | 16 x 16| 0.079  | 0.000  | 1.001   | 0.020   | 0.960         | 0.961| 954      | 938  | 907| 1.028    |
|       | 32 x 32| 0.079  | 0.000  | 1.001   | 0.005   | 0.960         | 0.960| 954      | 938  | 907| 1.022    |

TABLE I. NUMERICAL RESULTS OF ALGORITHM 1 ON STOKES EQUATION.
4. CONCLUSIONS

In this paper, we apply power method to computing the spectral radius of the saddle point matrices, and propose an effective priori indicator to predict the convergence rate. The present numerical results are in line with our expectations. However, we have to note that the discussion in this article relies on some approximate calculation. Further rigorous theoretical analysis and more numerical experiments are needed to verify the reliability of the new proposition.

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