A Survey on Wide Diameter of BC Networks

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Abstract. Wide diameter is an important parameter to measure the performance of network parallel communication. The hypercube is one of the most popular interconnection networks. The BC networks consist of the hypercube and most of the hypercube variations. In this paper, we give a survey of the research results on wide diameter of BC networks.

Introduction

A multiprocessor system can be modeled as a simple graph \( G = (V, E) \), where \( V \) corresponds to the processor set and \( E \) corresponds to the communication link set. In this paper, the terms network and graph, node and vertex, link and edge are used interchangeably. Two nodes, \( x \) and \( y \), are adjacent if \( x \) and \( y \) are connected by an edge. A path of a graph \( G \) is a sequence of distinct nodes, which can be represented by \( x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n \), where \( x_0 \) is the beginning node and \( x_n \) is the ending node. The length of a path \( P \) is the number of edges in \( P \), which can be denoted by \( \ell(P) \). The distance between two nodes \( x \) and \( y \) in graph \( G \) is the length of the shortest path connecting them, which can be denoted by \( d_G(x, y) \).

The diameter of graph \( G \), denoted by \( \text{diam}(G) \), is defined as the maximal of \( d_G(x, y) \) among all pairs of distinct nodes in \( G \). That is, \( \text{diam}(G) = \max\{d_G(x, y)|x, y \in V(G)\} \).

Two paths between \( x \) and \( y \) are internal node disjoint if there are no common nodes in these two paths except \( x \) and \( y \). By Menger’s theorem\(^1\), there are \( k \) internal disjoint paths between any pair of nodes \( x \) and \( y \) in \( G \) if graph \( G \) is \( k \)-connected. Let \( C(x, y) \) denote a set of \( k \) internal disjoint paths \( \{P_1, P_2, \ldots, P_k\} \) between \( x \) and \( y \). Then, \( C(x, y) \) is called a \( k \)-container between \( x \) and \( y \) in graph \( G \). The length of \( C(x, y) \), denoted by \( \ell(C(x, y)) \), is \( \max\{\ell(P_i)|1 \leq i \leq k\} \). The wide distance between \( x \) and \( y \), denoted by \( d_k(x, y) \), is \( \min\{\ell(C(x, y))\} \). The wide diameter of graph \( G \), denoted by \( D_k(G) \), is \( \max\{d_k(x, y)|x, y \in (G)\} \)\(^2\).

The wide diameter parameter of networks is proposed by Hsu\(^3\). When designing a network topology, the fault tolerance property of the network should be considered. The wide diameter can be used to evaluate the multipath communication performance of networks\(^4\). Therefor, researchers paid much attention to this parameter\(^5, 6, 7\).

The \( n \)-dimensional hypercube \( Q_n \) is a regular graph with recursive structure. An \( n \)-dimensional hypercube \( Qn \) has \( 2n \) nodes and \( n2^{n-1} \) edges. Owing to many advantageous properties, \( Q_n \) is one of the most popular interconnection networks in multiprocessor systems. However, \( Q_n \) still has some shortcomings such as large diameter and large connectivity. Hence, researchers proposed some improved structures of hypercube by changing some edges. Hypercube variations such as m\( \phi \)bius cube, crossed cube and twisted cube have smaller diameters which are about a half of that of \( Q_n \). These hypercube variations have two properties in common with hypercubes, which are recursive constructability and bijective connection. To study these networks in a unified way, a class of graphs, called bijective connection graphs\((BC \text{ networks for short})\), are introduced based on these two properties. The hypercube and the hypercube variations mentioned above are BC networks.
Researchers have done a lot of work on wide diameter of hypercubes and some hypercube variations. In this paper, we will give a survey on wide diameter of BC networks.

The Wide Diameter of BC Networks

Hypercubes

The $n$-dimensional hypercube $Q_n$ has $2^n$ vertices, each of which is labeled by an $n$-bit binary number $x_1x_2...x_n$, where $x_i \in \{0, 1\}$ and $1 \leq i \leq n$. Two vertices $x$ and $y$ are adjacent if and only if their binary numbers differ in only one bit. Saad and Schultz [9] proved that the diameter $diam(Q_n)$ of hypercubes is $n$, and the results of the wide diameter of hypercubes are also given in [9].

$$D_k(Q_n) = \begin{cases} n & \text{if } 1 \leq k \leq n-1, \\ n+1 & \text{if } k=n. \end{cases}$$

Möbius Cubes

In [10], Cull and Larson proposed the $n$-dimensional Möbius cube which is denoted by $MQ_n$. $MQ_n$ also has $2n$ vertices labeled by $x_1x_2...x_n$, the node $X = x_1x_2...x_n$ is connected to $n$ neighbors $Y_i(1 \leq i \leq n)$, where each $Y_i$ satisfies the following equation:

$$Y_i = x_1x_2...x_{i-1}x_{i+1}...x_n \text{ if } x_{i-1} = 0 \text{ or } x_{i+1}...x_n \text{ if } x_{i-1} = 1.$$  

$x_0$ is not defined in the equation when $i = 1$, so we can assume that $x_0 = 1$ or $x_1 = 0$. If we assume that $x_0 = 1$, $MQ_n$ is called a “1-Möbius cube”; if we assume that $x_0 = 0$, $MQ_n$ is called a “0-Möbius cube”. The diameter of the Möbius cube is about a half of that of the hypercube. If $MQ_n$ is a 1-Möbius cube, $diam(MQ_n) = [(n + 1)/2]$ where $n \geq 1$; if $MQ_n$ is a 0-Möbius cube, $diam(MQ_n) = [n + 1/2]$ where $n \geq 4$. Xu and Deng [11] showed the result of the wide diameter of Möbius cubes.

$$D_n(MQ_n) \leq [(n+1)/2] + 2$$

Crossed Cubes

In [12], Efe proposed the crossed cube denoted by $CQ_n$, which can be recursively defined as follows.

$CQ_1$ is a complete graph with two vertices labeled 0 and 1. $CQ_n$ consists of $CQ^0_{n-1}$ and $CQ^1_{n-1}$. The labels of vertices in $CQ^0_{n-1}$ (respectively, $CQ^1_{n-1}$) are prefixed by 0(respectively, 1). Let $R = \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. The vertex $X = 0x_{n-2}...x_0 \in V(CQ^0_{n-1})$ and $Y = 1y_{n-2}...y_0 \in V(CQ^1_{n-1})$ are adjacent if and only if

1. $x_{n-2} = y_{n-2}$ if $n$ is even, and
2. $x_{2i+1}x_{2i+2}, y_{2i+1}y_{2i+2} \in R$ for all $0 \leq i \leq [n + 1/2]$.

The diameter of $CQ_n$ is $[(n + 1)/2]$. In [13], Chang et al. showed the wide diameter of the crossed cube as follows.

$$D_n(CQ_n) \leq [(n+1)/2] + 2$$

Twisted Cubes

In [14], Hilbers et al. proposed the twisted cube which is denoted by $TQ_n$. However, the authors only consider the case that $n$ is odd. $TQ_n$ can be defined recursively as follows: $TQ_1$ is a complete graph with two vertices 0 and 1. $TQ_n$ can be decomposed into four subgraphs $S^0_{0,0}$, $S^0_{0,1}$, $S^1_{0,0}$ and $S^1_{0,1}$. Subgraph $S^0_{j}$, which is isomorphic to $TQ^{n-2}$, contain vertices $x = x_{n-3}x_{n-2}...x_0$ where $x_{n-1} = i$ and $x_{n-2} = j$. Let $Pi(x) = x_i \oplus x_{i-1} \oplus ... \oplus x_0$ be a parity function, where $\oplus$ is the exclusive-or operation. If $P_{n-3}(x)$
= 1, vertex \( x = x_{n-1}x_{n-2}...x_0 \) is connected to \( \overline{x_{n-1}}x_{n-2}x_{n-3}...x_0 \) and \( \overline{x_{n-1}x_{n-2}}x_{n-3}...x_0 \). And if \( P_{n-3}(x) = 0 \), vertex \( x = x_{n-1}x_{n-2}...x_0 \) is connected to \( x_{n-1}\overline{x_{n-2}}x_{n-3}...x_0 \) and \( \overline{x_{n-1}x_{n-2}}x_{n-3}...x_0 \).

The diameter of the twisted cube is \( \lceil (n + 1)/2 \rceil \). In [15], Chang et al. determined the wide diameter of the crossed cube.

\[
D_n(TQ_n) \leq \lceil (n/2) \rceil + 2
\]

**BC Networks**

The BC networks consist of hypercube and most of the hypercube variants. The BC networks are defined as follows.

\[
\mathcal{H}_0 = \{ K_1 \} \text{ and } \mathcal{H}_n = \{ G_0 \oplus_M G_1 | G_0, G_1 \in \mathcal{H}_{n-1} \} \text{ where } M = \{ (x, \phi(x)) | x \in V(G_0), \phi(x) \in V(G_1), \text{ and } \phi \text{ is a bijection} \} \text{ and } G_0 \oplus_M G_1 = (V(G_0) \cup V(G_1), E(G_0) \cup E(G_1) \cup M).
\]

The upper bound of the wide diameter can be obtained from the heights of independent spanning trees (ISTs). In [16], Yang et al. presented an algorithm to build independent spanning trees (ISTs) for enhanced hypercubes. They claimed that the upper bound of wide diameter can be obtained by the maximum height of the ISTs for vertex symmetric graphs. But, there is no proof provided in the paper. In [17], Chang et al. proved the more general results, which can be applied to arbitrary graphs.

**Lemma 1.** Let \( \mu \) be any vertex in graph \( G \) and \( T_\mu = \{ T_1, T_2, ..., T_k \} \) be a set of ISTs, where \( k = \kappa(G) \).

Then,

\[
D_k(G) \leq \max_{\mu \in V(G)} \hat{h}(T_\mu)
\]

Where \( \hat{h}(T_\mu) \) is the maximum height of a tree in \( T_\mu \).

In [18], Cheng et al. presented a parallel algorithm to construct \( n \) ISTs which are rooted at an arbitrary vertex in conditional BC networks. They proved that the heights of the constructed ISTs are \( n + 1 \). Cheng et al. also proposed a parallel algorithm to construct ISTs in arbitrary BC networks where the heights of constructed ISTs are \( n+1 \). By Lemma 1, we have the following result.

\[
D_n(TQ_n) \leq n + 1
\]

**Summary**

The wide diameter can be used to evaluate the multipath communication performance of networks [4]. The BC networks consist of the hypercube and most of the hypercube variations. In this paper, we review the research results on wide diameter of some of BC networks. By analyzing the heights of independent spanning trees, we obtain the upper bound of wide diameter of BC networks.

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