Calculating Floquet Multipliers for Periodic Solution of Non-smooth Dynamical System

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Abstract. In this paper, we will discuss how to calculate Floquet multipliers of non-smooth dynamical systems which are continuous or discontinuous. The saltation matrix of discontinuous system plays an important role when computing the Floquet multipliers. In addition, a special periodic solution, semi-limit cycle, will be presented. We find that the stability of this periodic solution cannot be analyzed by the classical Floquet theory. More investigation is needed to enrich the theory of Floquet multipliers to non-smooth systems.

Introduction

Floquet theory is the mathematical theory periodic systems of ordinary differential equations (ODEs) [1]. For a long period of time, the Floquet multiplier theory has been an indispensable tool in the stability and bifurcation analysis for periodic responses [2-5]. This theory, however, was developed based on the assumption of system smoothness. As the study of non-smooth dynamical systems has increasingly attracted a huge amount of research interest, it has been an urgent to study the Floquet theory in non-smooth dynamical system.

In this paper, we will discuss how to calculate Floquet multipliers of non-smooth dynamical systems which are continuous or discontinuous. The saltation matrix of discontinuous system will also be introduced, which plays an important role when computing the Floquet multipliers. In addition, a special periodic solution, semi-limit cycle, will be presented. We find that the stability of this periodic solution cannot be analyzed by the classical Floquet theory. More investigation is needed to enrich the theory of Floquet multipliers to non-smooth systems.

To introduce the Floquet theory, first we consider a smooth autonomous system

\[ x' = f(x). \] (1)

where the superscript denotes the differentiation with respect to time \( t \). The \( n \)-dimensional vector, \( f(x) \), is differentiable to the first-order at least. A periodic solution of (1) is a closed trajectory, i.e.,

\[ \exists T \in R_0^+ , x(T) = x(0). \] (2)

The smallest value of \( T \) (\( T > 0 \)) is called the period. Let \( \varphi_t(x(0)) \) denote the solution of (1) at time \( t \) starting from \( x(0) \) at time \( t = 0 \). The stability of the periodic orbit is determined by the eigenvalues of the monodromy matrix

\[ M = \frac{\partial \varphi_T(x)}{\partial x} \bigg|_{x=x(0)} \] (3)

whose eigenvalues are called Floquet multipliers. The monodromy matrix is the solution at time \( T \) of the variational equation [6]
\[
\frac{dM(t)}{dt} = \left. \frac{\partial f(x)}{\partial x} \right|_{x(t)} M(t), \ M(0) = I
\]  
(4)

The Floquet multipliers are independent of the choice of the point \( x(0) \) on the periodic solution. For the autonomous system, one of the Floquet multipliers is always 1. This Floquet multiplier is also called the trivial Floquet multiplier. A periodic solution is asymptotically stable if all Floquet multipliers except the trivial Floquet multiplier are strictly smaller than one in modulus [7]. Bifurcations occur when one or more Floquet multipliers cross the unit circle in the complex plane.

**Non-smooth Dynamical Systems**

Over the past decades, non-smooth dynamical systems have received a large amount of research interests of researchers from various fields [8-11]. Non-smooth dynamical systems can be divided into three types according to their degree of non-smoothness [12]:

1. Non-smooth continuous systems with a discontinuous Jacobian, like systems with purely elastic one-sided supports.
2. Systems described by differential equations with a discontinuous right-hand side, also called Filippov systems. The vector field of those systems is discontinuous. Examples are systems with visco-elastic supports and dry friction.
3. Systems which expose discontinuities (or jumps) in the state, like impacting systems with velocity reversals.

A wide variety of systems in biology, physics, and engineering are described by piecewise linear dynamical system [13-16]

\[
x' = A_i x + B_i, \ x \in V_i \subset R^n
\]  
(5)

where \( A_i, B_i, i = 1,2, \ldots m \) are the constant matrices and vectors respectively, \( V_i, i = 1,2, \ldots n \) are open non-overlapping regions separated by \( (n-1) \) dimensional sub-manifolds or called as boundaries. The system (5) has switching boundaries \( \Sigma = \{x \in R^n | h_i(x) = 0\} \), where \( h_i(x) = 0, i = 1,2, \ldots m \) are the hyper-surface. Many of the three types of systems mentioned above can be represented by such system. This piecewise linear system can be continuous or discontinuous at right-hand side.

**Saltation Matrix of Non-smooth Systems**

Consider the nonlinear system with discontinuous right-hand side

\[
x' = f(x,t) = \begin{cases} f_-(x,t), & x \in V_- \\ f_+(x,t), & x \in V_+ \end{cases}
\]  
(6)

with the initial condition

\[
x(t = 0) = x_0
\]  
(7)

The state-space \( R^n \) is split into two subspaces \( V_- \) and \( V_+ \) by a hyper-surface \( \Sigma = \{x \in R^n | h(x(t)) = 0\} \), where \( h(x(t)) = 0 \) is a scalar indicator function. The right-hand side \( f(x,t) \) is assumed to be discontinuous on \( \Sigma \) but is piecewise continuous and smooth on \( V_- \) and \( V_+ \). Discontinuous systems exhibit discontinuities in the time evolution of the fundamental solution matrix [17]. The jumps occur when the trajectory crosses a switching boundary, as \( f(x,t) \) is discontinuous on the switching boundary. The jump can be described by a saltation matrix \( S \)

\[
M(t_{p+}, t_0) = SM(t_{p-}, t_0)
\]  
(8)

where \( M(t_{p-}, t_0) \) is the fundamental solution matrix before the jump and \( M(t_{p+}, t_0) \) after the jump which occurs at \( t = t_p \). The saltation matrix \( S \) can be expressed as
\[ S = I + \frac{(f_p-f_{-p})n^T}{n^T f_p - \frac{\partial h}{\partial x}(t_p x(t_p))} \tag{9} \]

where \( n \) is the normal to the switching boundary

\[ n = n(x(t), t) = \text{grad}(h(x(t), t)) \tag{10} \]

**Calculation of Floquet Multipliers to Non-smooth Systems**

To introduce how to calculate Floquet multipliers for periodic solution of system (5) for simplicity, we first consider the following system with only one boundary

\[ \Sigma \equiv h(x_2) = 0 \]

\[ x' = \begin{cases} A_1 x + B_1, & x \in V_1 \\ A_2 x + B_2, & x \in V_2 \end{cases} \tag{11} \]

Assume that system (11) has a periodic solution \( x = \varphi(t, x_0) \). Without loss of generality, we assume the initial state \( x_0 \) of the periodic solution is located at the region \( V_1 \), as is shown in Figure 1. The periodic curve crosses the boundary at \( x_1 \) for the first time after \( t_1 \) and at \( x_2 \) for the second time after \( t_2 \), then it goes back to \( x_0 \) after \( T \). Therefore, we obtain the piecewise smooth variation equation at different regions

\[ \Delta x' = \begin{cases} A_1 \Delta x, & t \in (0, t_1) \cup (t_2, T) \\ A_2 \Delta x, & t \in (t_1, t_2) \end{cases} \tag{12} \]

Integrating Eq. (12) over \((0, t_1)\) under the same initial condition utilized for Eq. (1.4), we can obtain a matrix \( M_1 \). If the system (11) is discontinuous at right-hand side, due to the jump of the fundamental solution matrix, the matrix \( M_1 \) should be updated as \( M_1 := S_1 M_1 \) with \( S_1 \) as the saltation matrix associated with the intersection point \( x_1 \). Take \( S_1 M_1 \) as the updated initial conditions and integrate (12) over \((t_1, t_2)\), we obtain a matrix \( M_2 \) which is further updated as \( M_2 := S_2 M_2 \), with \( S_2 \) as the saltation matrix associated with the intersection point \( x_2 \). Finally, we obtain the monodromy matrix at \( t = T \) by integrating (12) over \((t_2, T)\) with the updated initial conditions as \( M_2 \). Note that the fundamental solution of \( x' = Ax \) is \( x = e^{At} \), where \( e^A \) is called as state transition matrix and can be computed easily by common mathematical software. The monodromy matrix of system (11) is

\[ M = e^{A_1(T-t_2)} S_2 e^{A_2(t_2-t_1)} S_1 e^{A_1 t_1} \tag{13} \]

where \( S_1, S_2 \) are the saltation matrices. According to Eq. (9) we have

\[ S_1 = I + \frac{(A_2-A_1)x_1+B_2-B_1)n^T}{n^T(A_1x_1+B_1)}, \quad S_2 = I + \frac{(A_2-A_1)x_2+B_1-B_2)n^T}{n^T(A_2x_2+B_2)} \tag{14} \]

The eigenvalues of \( M \) are the Floquet multipliers of system (11). If system (11) is continuous at right-hand side, then \( S_1 = S_2 = I \). In this case, the monodromy matrix of system (11) can be rewritten as

\[ M = e^{A_1(T-t_2)} e^{A_2(t_2-t_1)} e^{A_1 t_1} \tag{15} \]
Figure 1. Periodic orbit of system (11) with two sub-regions.

Figure 2. Periodic orbit of system (5) with three sub-regions.

With the above discussion, the monodromy matrix of system (5) with multiple boundaries can be deduced similarly. For instance, Figure 2 shows a periodic solution of system (11) with three regions. Assume that the initial state $x_0$ of the periodic solution is located at the region $V_1$. The periodic curve crosses the boundary at $x_1, ..., x_4$ for the time $t_1, ..., t_4$, respectively. From Figure 2 we can see that the fundamental solution matrix jumps four times. The monodromy matrix of system (5) with three regions is

$$M = e^{A_1(T-t_4)}S_4e^{A_2(t_4-t_3)}S_3e^{A_3(t_3-t_2)}S_2e^{A_2(t_2-t_1)}S_1e^{A_1 t_1}$$

(16)
where \( S_1, \ldots S_4 \) are the saltation matrices. If system (5) is continuous at right-hand side, Eq. (16) can be rewritten as

\[
M = e^{A_1(T-t_4)}e^{A_2(t_4-t_3)}e^{A_3(t_3-t_2)}e^{A_2(t_2-t_1)}e^{A_1t_1}
\]  

(17)

The eigenvalues of \( M \) are the Floquet multipliers of system (5). As this system is piecewise linear, it can be solve analytically or numerically at each region by giving an initial condition. Therefore the travel time \( t_1, \ldots t_n \) and the intersection point \( x_1, \ldots x_n \) can be easily obtained.

For a system (5) with more switching boundaries, assume that the periodic orbit cross though each sub-region, the monodromy matrix can be deduced as

\[
M = e^{A_1(T-t_n)}S_ne^{A_n(t_n-t_{n-1})} \cdots S_2e^{A_2(t_2-t_1)}S_1e^{A_1t_1}
\]

(18)

where \( t_1, \ldots t_n \) are the travel time at boundary, \( S_1, \ldots S_n \) are the saltation matrices, sequentially.

A Special Periodic Solution

In this section, we will introduce a special periodic solution which is tangent to the switching boundary. Consider a non-smooth continuous system

\[
\begin{align*}
x_1' &= x_2 \\
x_2' &= -x_1 - x_2 - |x_2 - 1|
\end{align*}
\]

(19)

with the switching boundary \( \Sigma = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 - 1 = 0\} \). It is a non-smooth continuous system and can be rewritten as

\[
\begin{align*}
x_1' &= A_1x + B_1, \quad x_2 \geq 1 \\
x_2' &= A_2x + B_2, \quad x_2 < 1
\end{align*}
\]

(20)

where \( A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \). The phase plane starting from a series of initial conditions of system (20) is shown in Figure 3. There is a critical periodic solution, inside which a series of closed orbits arise, and outside all phase trajectories are attracted by the critical orbit. This critical orbit is called semi-limit cycle. As it is tangent to the boundary, we can only obtain the variational equation of this periodic solution in the lower region

\[
\Delta x' = A_2\Delta x.
\]

(21)

The period of the semi-limit \( T = 6.28 \). If we integrate Eq. (21) over \((0,T)\), we can obtain the monodromy matrix

\[
M = e^{A_2T}
\]

(22)

The two Floquet multipliers are both equal to 1. It means that the semi-limit cycle is stable but not asymptotically stable, like simple harmonic vibration. However, this closed orbit does attract the outer phase trajectories. The semi-limit cycle is therefore asymptotically stable on the outside. Note that system (19) is autonomous, one of the multipliers is always equal to 1. The other multiplier can’t characterize the stability, whatever its value is, as the semi-limit cycle is asymptotically stable outside but not asymptotically stable inside.
From above discussion, the stability of semi-limit cycle can’t be analyzed by the classical Floquet theory. It is possibly unable to obtain the right variational equation when the periodic orbit is tangent to the switching boundary. More investigation are needed to enrich the theory of Floquet multipliers to non-smooth systems.

Conclusion

We have discussed how to calculate Floquet multipliers of non-smooth linear dynamical systems. The monodromy matrices of system with one boundary and multiple boundaries were deduced. The saltation matrix of discontinuous system plays an important role when computing the Floquet multipliers. In addition, a special periodic solution, semi-limit cycle, was be presented. From this system's phase plane, we can see that there is a critical periodic solution, inside which a series of closed orbits arise, and outside all phase trajectories are attracted by the critical orbit. We find that the stability of this periodic solution can’t be analyzed by the classical Floquet multipliers. More investigation are needed to enrich the theory of Floquet multipliers to non-smooth systems.

References


