Existence and Uniqueness of Fractional Differential Equational Involving Katugampola Derivative

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Abstract. In this paper, by using both Banach fixed point theorem and Schauder's fixed point theorem we obtain the existence and uniqueness results for a kind of Katugampola fractional differential equation.

Introduction

In recent year, due to the local limit definition of integral order ordinary differential equation or partial differential equation is not suitable to describe the history dependent process, fractional calculus has attracted more and more attention in many fields. Fractional order differential equations can describe objective laws and the essence of things more accurately than integral order differential equations. Leibnitz discovered fractional derivatives in 1695. Many scholars have devoted themselves to the study of fractional calculus. Now the most commonly used fractional calculus definition in basic mathematics research and engineering application research are Riemann-Liouville calculus definition, Caputo differential definition, Grunwald-Letnikov differential definition etc [10]. Fractional dynamic model, score control system, fractional population dynamics model and fractional fluid mechanics involves at least one general or partial fractional derivative. Fractional differential equations are widely used in practice, and greatly enriched the content of mathematical theory, and penetrated into many fields of natural science.

In 2011, U. Katugampola presented a new fractional integration, which generalizes the Riemann-Liouville and Hadamard integral into a single form, which when a parameter fixed at different values, produces the above integrals as special cases [7]. In 2014, he presented two representations of the generalized derivative called Katugampola derivative [8]. In the same work, he obtained boundedness of generalized fractional integral in an extended Lebesgue measurable space and gave illustrative examples. In [9], he obtained existence and uniqueness results to the solution of initial value problem for a class of generalized fractional differential equations. Further research on its properties has an expansion formula, application of variation method, application of control theory, convexity, integral inequality and Hermite-Hadamard type inequalities for the generalized operator and similar operators [2-4, 6, 11-21].

In this paper, we initiate to study the existence and uniqueness of the following Katugampola fractional order differential equation of non-local conditions:

\[
\begin{align*}
(\rho D_{a+}^{\alpha} x)(t) &= f(a, x(t)), \quad t \in (a, h], \\
(\rho I_{a+}^{1-\alpha} x)(0) &= x_0 = \sum_{i=1}^{m} C_i x(t_i),
\end{align*}
\]

where $\rho D_{a+}^{\alpha}$, $\rho I_{a+}^{1-\alpha}$ is the Katugampola fractional integral and derivative, which will be given in next section, $x \in \mathbb{R}^n$, $0 < \alpha < 1$, $\rho > 0$ and $a \in \mathbb{R}$, the state $x(t)$ takes value space $C$ with norm $\| \cdot \|$, $f : (a, h] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given functions satisfying some assumptions, $t_i (i=0,1,\cdots,m)$ satisfying
$a < t_1 \leq t_2 \leq \cdots \leq t_m < h$ and $c_i$ is real number, $x_0 \in \mathbb{R}^n$. Here nonlocal condition are more effective than the initial conditions $(^{(\rho)} I_{a+}^{1-\alpha} x)(0) = x_0$ in terms of physical problems.

The rest of the paper is organised as follow: in Section 2 we give some preliminary facts that we need in the subsequent. In Section 3 we present our main results on existence of considered problem and prove it with Schauder fixed point theorem. In Section 4 we prove uniqueness of considered problem with Banach fixed theorem.

Preliminaries

As usual, $C$ denotes the space of all continuous functions $x : (a, h) \to \mathbb{R}^n$ with the norm

$$
\| x \|_C = \sup_{t \in (a, h)} \| x(t) \|_E
$$

and $AC(a, h)$ be the space of absolutely continuous functions from $(a, h)$ into $\mathbb{R}^n$.

Here we define the weighted space of measurable functions $x$, by

$$
C_{\alpha, \rho}(a, h) = \left\{ x \in C(a, h) \left| \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-\alpha} x(t) \in C(a, h) \right. \right\}, \quad \alpha > 0,
$$

with the norm

$$
\| x \|_{C_{\alpha, \rho}} = \sup_{t \in (a, h)} \left\| \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-\alpha} x(t) \right\|.
$$

Consider the space $X_{\rho}^\alpha(a, b)(c \in \mathbb{R}, 1 \leq p \leq \infty)$ of those measurable functions $f$ on $[a, b]$ for which $\| f \|_{X_{\rho}^\alpha(a, b)} < \infty$, with the norm

$$
\| f \|_{X_{\rho}^\alpha(a, b)} = (\int_{a}^{b} | t^\rho f(t) |^{1/p} \frac{dt}{t})^{1/p} < \infty.
$$

**Definition 2.1.** [7] The Katugampola left-sided fractional integral of order $\alpha > 0$, $\rho > 0$ of $f \in X_{\rho}^\alpha(a, b)$ for $-\infty < a < x < \infty$, is defined by

$$
\left( ^{\rho} I_{a+}^{\alpha} f \right)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{a}^{x} \frac{\tau^{\rho-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau.
$$

(2)

**Definition 2.2.** [8] The Katugampola fractional derivative, associated with the generalized fractional integrals (2) are defined, for $0 \leq a < x < \infty$, $n = [\alpha]+1$, by

$$
\left( ^{\rho} D_{a+}^{\alpha} f \right)(x) = \left( x^{1-\rho} \frac{d}{dx} \right)^n \left( ^{\rho} I_{a+}^{\alpha-n} f \right)(x)
$$

$$
= \frac{\rho^{\alpha-n+1}}{\Gamma(n - \alpha)} \left( x^{1-\rho} \frac{d}{dx} \right)^n \int_{a}^{x} \frac{\tau^{\rho-1}}{(x^\rho - \tau^\rho)^{\alpha-n+1}} f(\tau) d\tau.
$$

(3)

**Lemma 2.3.** [14] According to (2), we have

$$
^{\rho} I_{a+}^{\alpha} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\alpha+\beta-1}, \quad \alpha \geq 0, \beta > 0.
$$
Theorem 2.4. [5] [Schauder fixed point theorem] Let $E$ be a Banach space and $Q$ be a nonempty bounded convex and closed subset of $E$ and $\Lambda : Q \to Q$ is compact, and continuous map. Then $\Lambda$ has at least one fixed point in $Q$.

Theorem 2.5. [5] [Arzela-Ascoli theorem] A subset $F$ of $C(X)$ is relatively compact if and only if it is closed, bounded and equicontinuous.

Theorem 2.6. [1] [Banach fixed point theorem] Let $(x,d)$ be a metric space. Then a map $T : X \to X$ is called a contraction mapping on $X$ if there exists $q \in (0,1)$ such that

$$d(T(x), T(y)) \leq qd(x, y)$$

for all $x, y$ in $X$.

Existence of Solutions

In this section, by using the Ascoli-Arzela Theorem and Schauder fixed point theorem, we will obtain some sufficient conditions ensuring the existence of solution.

Lemma 3.1. Let $f : (a, h] \times \mathbb{R}^n \to \mathbb{R}^n$ be measurable function. A solution of the problem (1) can be expressed as

$$x(t) = \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \frac{1}{\rho^{\alpha-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=1}^{m} C_i (t_i^\rho - a^\rho)^{\alpha-1}} \sum_{i=1}^{m} C_i \int_{a}^{t_i} \frac{\tau^{\rho-1}}{(t_i^\rho - \tau^\rho)^{1-\alpha}} f(\tau, x(\tau)) d\tau$$

$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{a}^{t} \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau, x(\tau)) d\tau,$$

where $0 < \alpha < 1$, $\rho > 0$, and for all $C_i, t_i, \rho^{\alpha-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=1}^{m} C_i (t_i^\rho - a^\rho)^{\alpha-1} \neq 0$. Then, we can conclude that (4) is the solution of problem (1).

Theorem 3.2. (Existence) Let $\alpha > 0$, the function $t \mapsto f(t,x)$ is measurable on $(a,h]$ for each $x \in C_{\alpha, \rho}$, and the function $x \mapsto f(t,x)$ is continuous on $C_{\alpha, \rho}$ for a.e. $t \in (a,h]$, further satisfy $|f(t, x(t))| \leq l(t)$, $l(t) \in L^2$ and non-negative. Then the problem (1) has at least one solution.

Proof. If $l(t)=0$, $f(t,x)=0$ for all $t \in (a,h]$, in this case it is clear by direct substitution that the function $x(t) : (a,h] \to \mathbb{R}^n$ with $x(t)=0$ is a solution of the problem (1). Hence a solution exists in this case.

For $l(t) \neq 0$, we apply Lemma 3.1 that problem (1) is equivalent to (4). We consider the operator $\Lambda : C_{\alpha, \rho} \to C_{\alpha, \rho}$ such that

$$(\Lambda x)(t) = \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \frac{1}{\rho^{\alpha-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=1}^{m} C_i (t_i^\rho - a^\rho)^{\alpha-1}} \sum_{i=1}^{m} C_i \int_{a}^{t_i} \frac{\tau^{\rho-1}}{(t_i^\rho - \tau^\rho)^{1-\alpha}} f(\tau, x(\tau)) d\tau$$

$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{a}^{t} \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau, x(\tau)) d\tau.$$

Clearly, the fixed points of this operator equation $\Lambda x(t) = x(t)$ are solutions of the problem (1).

For any $x \in C_{\alpha, \rho}$ and each $t \in (a,h]$ \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

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\[
\left| \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-a} (\Lambda x)(t) \right| \\
\leq \frac{1}{\rho^{a-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=0}^{m-1} C_i \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-a}} \left| \sum_{i=1}^{m} C_i \int_{\tau}^{t^\rho} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-a} f(\tau, x(\tau)) \, d\tau \right| + \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-a} \Gamma(\alpha) \int_{\tau}^{t^\rho} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-a} f(\tau, x(\tau)) \, d\tau \\
\leq \frac{1}{\rho^{a-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=0}^{m-1} C_i \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-a}} \left| \sum_{i=1}^{m} C_i \int_{\tau}^{t^\rho} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-a} f(\tau, x(\tau)) \, d\tau \right| \| f \|_{\infty} + \frac{h^\rho - a^\rho}{\rho \Gamma(1+\alpha)} \| f \|_{\infty} := K.
\]

Thus

\[
\| \Lambda x \|_{C_{a,\rho}} \leq K.
\]

Thus, \( \Lambda \) transforms the ball \( B_k = \{ z \in C_{a,\rho} : \| z \|_{C} \leq K \} \) into itself. We shall show that the operator \( \Lambda : B_k \rightarrow B_k \) satisfies all the conditions of the Schauder's Second Fixed Theorem. The proof is given in following several steps.

Step 1: \( \Lambda : B_k \rightarrow B_k \) is continuous.

Let \( \{ x_n \} \), \( n = 1, 2, \cdots \), be a sequence such that \( x_n \rightarrow x \) in \( B_k \). Then, for each \( t \in (a, h) \), by Lebesgue dominated convergence theorem, we have

\[
\left| (t^\rho - a^\rho) \left( \Lambda x_n \right)(t) - (t^\rho - a^\rho) \left( \Lambda x \right)(t) \right| \\
\leq \left| \frac{1}{\rho^{a-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=0}^{m-1} C_i \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-a}} \left| \sum_{i=1}^{m} C_i \int_{t_1}^{t} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-a} f(\tau, x(\tau)) \, d\tau \right| \right|_{\infty} + \frac{h^\rho - a^\rho}{\rho \Gamma(1+\alpha)} \| f \|_{\infty} \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

by \( f \) is continuous of \( x \) and Lebesgue Control Convergence Theorem, it is obvious that \( \| (\Lambda x_n) - (\Lambda x) \|_{C_{a,\rho}} \rightarrow 0, \text{ as } n \rightarrow \infty \).

Step 2: \( \Lambda(B_k) \) is uniformly bounded.

Since \( \Lambda(B_k) \subset B_k \) and \( B_k \) is bounded. Hence, \( \Lambda(B_k) \) is uniformly bounded.

Step 3: \( \Lambda(B_k) \) is equicontinuous.

Let \( 0 \leq t_1 \leq t_2 \leq h \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( x \in B_k \). We have
\[
\left|\left(\frac{t^\varrho - a^\varrho}{\rho}\right)^{-a} (\Lambda x)(t_1) - \left(\frac{t^\varrho - a^\varrho}{\rho}\right)^{-a} (\Lambda x)(t_2)\right|
\]

\[=rac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left((t_1^\varrho - a^\varrho)^{1-a} \varphi(t_1^\varrho - \tau^\varrho)^{\alpha-1} - (t_2^\varrho - a^\varrho)^{1-a} \varphi(t_2^\varrho - \tau^\varrho)^{\alpha-1}\right) \varphi(\tau, x(\tau)) d\tau
\]

\[+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2^\varrho - a^\varrho)^{1-a} (t_2^\varrho - \tau^\varrho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau
\]

\[\leq \frac{1}{\Gamma(\alpha)} \left\{ \left[ \int_{t_1}^{t_2} \left((t_1^\varrho - a^\varrho)^{1-a} \varphi(t_1^\varrho - \tau^\varrho)^{\alpha-1} - (t_2^\varrho - a^\varrho)^{1-a} \varphi(t_2^\varrho - \tau^\varrho)^{\alpha-1}\right) \varphi(\tau, x(\tau)) d\tau\right]^\frac{1}{p}
\]

\[\times \left[ \int_{t_1}^{t_2} \varphi(\tau) d\tau\right]^\frac{1}{q} + \left[ \int_{t_1}^{t_2} (t_2^\varrho - a^\varrho)^{1-a} (t_2^\varrho - \tau^\varrho)^{\alpha-1} \varphi(\tau, x(\tau)) d\tau\right]^\frac{1}{p} \left[ \int_{t_1}^{t_2} \varphi(\tau) d\tau\right]^\frac{1}{q}\}

\[\leq \frac{1}{\Gamma(\alpha)} \left\{ \left[ \int_{t_1}^{t_2} \left((t_1^\varrho - a^\varrho)^{1-a} \varphi(t_1^\varrho - \tau^\varrho)^{\alpha-1} - (t_2^\varrho - a^\varrho)^{1-a} \varphi(t_2^\varrho - \tau^\varrho)^{\alpha-1}\right) \varphi(\tau, x(\tau)) d\tau\right]^\frac{1}{p} \times \|x\|_p
\]

\[+ \frac{1}{\rho \alpha} (t_2^\varrho - a^\varrho)^{1-a} (t_2^\varrho - t_1^\varrho)^{\alpha} \|x\|_p \right\}
\]

In either case, the expression on the right-hand side of (6) converges to 0 as \( t_2 \to t_1 \), which proves that \( \Lambda x \) is equicontinuous.

As a consequence of Steps 1 to 3 together with Arzela-Ascoli Theorem, we can conclude that \( \Lambda \) is continuous and compact. By applying the Schauder fixed point theorem, we conclude that \( \Lambda \) has a fixed point \( x \) which is a solution of the problem (1).

**Existence and Uniqueness of Solutions**

In this section, by using the Banach fixed point theorem, we will prove the existence and uniqueness of the solution for the problem (1).

For computational convenience, we give

\[
A = (-1)^{1-a} \rho^{-a} \beta(\alpha, 1-2\alpha) \left(\frac{(t^\varrho - a^\varrho)^\alpha}{\rho^{a-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=1}^{n} C_i (t_i^\varrho - a^\varrho)^{\alpha-1}}\right) \sum_{i=1}^{n} C_i \|x\|_p + \frac{h^\varrho - a^\varrho}{\Gamma(\alpha)}.
\]

(8)

**Theorem 4.1.** (Existence and Uniqueness) Assume there exists a constant \( L > 0 \) such that

\[|f(t, u) - f(t, v)| \leq L |u - v|, \text{ for each } t \in (a, h] \text{ and } u, v \in C_{a, \rho}, \text{ if}
\]

\[L A < 1,
\]

(9)

where \( A \) is defined by (8), the problem (1) has a unique solution on \( G \).

Proof. According to theorem 3.2, we know that \( \Lambda x \in C_{a, \rho} \) if \( x \in C_{a, \rho} \), i.e. \( \Lambda \) maps is into itself. We just only need to show that the operator \( \Lambda \) is a contraction, let \( u, v \in C_{a, \rho} \). Then, for each \( t \in [0, h] \), we obtain
\[ \left| \frac{(t^p - a^p)}{\rho} \right|^{-\alpha} (\Lambda u(t)) - \left( \frac{(t^p - a^p)}{\rho} \right)^{-\alpha} (\Lambda v(t)) \right| \]

\[ \leq \frac{1}{\rho^{\alpha-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=1}^{m} C_i (t_i^p - a^p)^{\alpha-1}} \left| \sum_{i=1}^{m} C_i \int_{t_i}^{t^p} \frac{\tau^{\alpha-1}}{(t_i^p - \tau)^{\alpha-1}} f(\tau, u(\tau)) - f(\tau, v(\tau)) \, d\tau \right| \]

\[ + \frac{(t^p - a^p)^{-\alpha}}{\Gamma(\alpha)} \int_{t_i}^{t^p} \frac{\tau^{\alpha-1}}{(t_i^p - \tau)^{\alpha-1}} f(\tau, u(\tau)) - f(\tau, v(\tau)) \, d\tau \]

\[ \leq L \| u - v \|_{\alpha, \rho} \left[ \frac{1}{\rho^{\alpha-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=1}^{m} C_i (t_i^p - a^p)^{\alpha-1}} \left| \sum_{i=1}^{m} C_i \int_{t_i}^{t_i^p} \frac{\tau^{\alpha-1}}{(t_i^p - \tau)^{\alpha-1}} \left( \frac{(t_i^p - a^p)^{\alpha-1}}{\rho} \tau^{\alpha-1}\right) \right| \right] \]

\[ \times \int_{t_i}^{t_i^p} \frac{\tau^{\alpha-1}}{(t_i^p - \tau)^{\alpha-1}} \left( \frac{(t_i^p - a^p)^{\alpha-1}}{\rho} \tau^{\alpha-1}\right) \, d\tau + \frac{(t^p - a^p)^{-\alpha}}{\Gamma(\alpha)} \int_{t_i}^{t^p} \frac{\tau^{\alpha-1}}{(t^p - \tau)^{\alpha-1}} \left( \frac{(t^p - a^p)^{\alpha-1}}{\rho} \tau^{\alpha-1}\right) \, d\tau \]

\[ = L \| u - v \|_{\alpha, \rho} \left[ \frac{1}{\rho^{\alpha-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=1}^{m} C_i (t_i^p - a^p)^{\alpha-1}} \left| \sum_{i=1}^{m} C_i \int_{t_i}^{t_i^p} \frac{\tau^{\alpha-1}}{(t_i^p - \tau)^{\alpha-1}} \left( \frac{(t_i^p - a^p)^{\alpha-1}}{\rho} \tau^{\alpha-1}\right) \right| \right] \]

\[ \times \int_{t_i}^{t_i^p} \frac{\tau^{\alpha-1}}{(t_i^p - \tau)^{\alpha-1}} \left( \frac{(t_i^p - a^p)^{\alpha-1}}{\rho} \tau^{\alpha-1}\right) \, d\tau + \frac{(t^p - a^p)^{-\alpha}}{\Gamma(\alpha)} \int_{t_i}^{t^p} \frac{\tau^{\alpha-1}}{(t^p - \tau)^{\alpha-1}} \left( \frac{(t^p - a^p)^{\alpha-1}}{\rho} \tau^{\alpha-1}\right) \, d\tau \]

\[ = L \| u - v \|_{\alpha, \rho} \left[ \frac{1}{\rho^{\alpha-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=1}^{m} C_i (t_i^p - a^p)^{\alpha-1}} \left| \sum_{i=1}^{m} C_i \int_{t_i}^{t_i^p} \frac{\tau^{\alpha-1}}{(t_i^p - \tau)^{\alpha-1}} \left( \frac{(t_i^p - a^p)^{\alpha-1} \mu^{\alpha-1}}{1 - \mu} \right) \right| \right] \]

\[ \times \int_{t_i}^{t_i^p} \frac{\tau^{\alpha-1}}{(t_i^p - \tau)^{\alpha-1}} \left( \frac{(t_i^p - a^p)^{\alpha-1} \mu^{\alpha-1}}{1 - \mu} \right) \, d\tau + \frac{(t^p - a^p)^{-\alpha}}{\Gamma(\alpha)} \int_{t_i}^{t^p} \frac{\tau^{\alpha-1}}{(t^p - \tau)^{\alpha-1}} \left( \frac{(t^p - a^p)^{\alpha-1} \mu^{\alpha-1}}{1 - \mu} \right) \, d\tau \]

\[ = L \| u - v \|_{\alpha, \rho} \left[ \frac{1}{\rho^{\alpha-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=1}^{m} C_i (t_i^p - a^p)^{\alpha-1}} \left| \sum_{i=1}^{m} C_i \int_{t_i}^{t_i^p} \frac{(t_i^p - a^p)^{\alpha-1} \mu^{\alpha-1}}{1 - \mu} \right| \right] \]

\[ \times \int_{t_i}^{t_i^p} \frac{(t_i^p - a^p)^{\alpha-1} \mu^{\alpha-1}}{1 - \mu} \, d\tau + \frac{(t^p - a^p)^{-\alpha}}{\Gamma(\alpha)} \int_{t_i}^{t^p} \frac{(t^p - a^p)^{\alpha-1} \mu^{\alpha-1}}{1 - \mu} \, d\tau \]

\[ = L \| u - v \|_{\alpha, \rho} \left[ \frac{1}{\rho^{\alpha-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=1}^{m} C_i (t_i^p - a^p)^{\alpha-1}} \left| \sum_{i=1}^{m} C_i \right| \right] \]

\[ + \frac{h^\alpha - a^\alpha}{\Gamma(\alpha)} \frac{1}{\rho^{\alpha-1} \Gamma(\alpha)^2 - \Gamma(\alpha) \sum_{i=1}^{m} C_i (t_i^p - a^p)^{\alpha-1}} \left| \sum_{i=1}^{m} C_i \right| \]

\[ = L A \| u - v \|_{\alpha, \rho} \cdot \]

Hence operator \( \Lambda \) is a contraction mapping and the operator \( \Lambda \) has a unique fixed point by Banach’s contraction principle, which corresponds to a unique solution of the problem (1). The proof is completed.

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