Various Higher-order Shear Deformation Theories for Vibration Analysis of Functionally Graded Plates

Li-jie ZHAO*, Ying-ying CHANG, Song Xiang and Meng-wei TIAN
Key Laboratory of General Aviation, Shenyang Aerospace University, Liaoning Shenyang 110136, China

Keywords: Vibration, Functionally graded plates, Higher-order theory, Analytical method.

Abstract. In the present paper, the four various higher-order shear deformation theories are used to analyze the free vibration of functionally graded plate. A navier-type analytical method is used to solve the governing differential equations[1]. Natural frequencies of simply supported functionally graded plates are calculated. The present results are compared with the available published results which verify the accuracy of various higher-order theories. The influences of side to thickness ratio and power law index on the fundamental frequencies of a simply supported square functionally graded plate are also studied.

Introduction
The material properties of the fiber-reinforced laminated composite materials are discontinuous across adjoining layers which result in the delaminating mode of failure. Functionally graded plates can overcome the delaminating mode due to their continuous variation of material properties from one surface to another[2]. The functionally graded material for high-temperature applications may be composed of ceramic and metal.

This paper uses various shear deformation theories of Touratier (1991), Mantari (2012), Karama (2003), Levinson (1980) to study the free vibration behavior of functionally graded plates. A navier-type analytical method is used to solve the governing differential equations. The present results are compared with those of Vel and Batra (2004) and Matsunaga (2008)[3,4].

Governing Equations and Boundary Conditions

Displacement Field
The displacement field of the higher order shear deformation theory is:

\[ U = u(x, y) - z \frac{\partial w(x, y)}{\partial x} + f(z)\phi_3(x, y) \]

\[ V = v(x, y) - z \frac{\partial w(x, y)}{\partial y} + f(z)\phi_4(x, y) \]

\[ W = w(x, y) \]

where \( u, v, w, \phi_3, \) and \( \phi_4 \) are the five unknown displacement functions of middle surface of the plate. \( h \) is the thickness of the plate.

The Transverse Shear Function
The transverse shear function in Touratier (1991) is:

\[ f(z) = \frac{h}{\pi} \sin\left(\frac{\pi z}{h}\right) \]

The transverse shear function in Mantari (2012) is:
\[ f(z) = \sin \left( \frac{\pi z}{h} \right) e^{\frac{1}{2} \frac{\pi z}{h}} + \frac{\pi}{2h} z \]  

The transverse shear function in Karama (2003) is:

\[ f(z) = ze^{-2z/v^2} \]  

(4)

The transverse shear function in Levinson (1980) is:

\[ f(z) = z(1 - \frac{4z^2}{3h^2}) \]  

(5)

**The Strain–Displacement Relationships**

The strain–displacement relationships can be expressed in the form of

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix} = \begin{bmatrix}
\dfrac{\partial U}{\partial x} \\
\dfrac{\partial V}{\partial y} \\
\dfrac{\partial U}{\partial y} + \dfrac{\partial V}{\partial x} \\
\dfrac{\partial W}{\partial z} + \dfrac{\partial V}{\partial y} \\
\dfrac{\partial W}{\partial z} + \dfrac{\partial U}{\partial x}
\end{bmatrix}
\]

(6)

By the principle of virtual displacements, we can obtain the following Euler–Lagrange equations[5]:

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1 \mu_{xy} + I_4 \phi_{x,yy} - I_2 \frac{\partial w}{\partial x}
\]

\[
\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = I_1 \nu_{xy} + I_4 \phi_{y,xx} - I_2 \frac{\partial w}{\partial y}
\]

\[
\frac{\partial^2 M_y}{\partial x^2} + \frac{\partial^2 M_x}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} = I_2 \frac{\partial u}{\partial x} + I_2 \frac{\partial v}{\partial y} + I_3 \frac{\partial^2 w}{\partial x^2} + I_3 \frac{\partial^2 w}{\partial y^2} + I_3 \left( \frac{\partial^2 w}{\partial x \partial y} \right)
\]

\[
\frac{\partial M_y}{\partial y} + \frac{\partial M_x}{\partial x} - Q_y^f = I_3 \nu_{xy} + I_6 \phi_{y,x} - I_5 \frac{\partial w}{\partial y}
\]

(7)

where
\[ N_x = A_{11} \frac{\partial u}{\partial x} + A_{12} \frac{\partial v}{\partial y} - B_{11} \frac{\partial^2 w}{\partial x^2} - B_{12} \frac{\partial^2 w}{\partial y^2} + E_{11} \frac{\partial \phi_x}{\partial x} + E_{12} \frac{\partial \phi_y}{\partial y} \]

\[ N_y = A_{12} \frac{\partial u}{\partial x} + A_{11} \frac{\partial v}{\partial y} - B_{12} \frac{\partial^2 w}{\partial x^2} - B_{11} \frac{\partial^2 w}{\partial y^2} + E_{12} \frac{\partial \phi_x}{\partial x} + E_{11} \frac{\partial \phi_y}{\partial y} \]

\[ N_{xy} = A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - 2B_{66} \frac{\partial^2 w}{\partial x \partial y} + E_{66} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \]

\[ M_x = B_{11} \frac{\partial u}{\partial x} + B_{12} \frac{\partial v}{\partial y} - D_{11} \frac{\partial^2 w}{\partial x^2} - D_{12} \frac{\partial^2 w}{\partial y^2} + F_{11} \frac{\partial \phi_x}{\partial x} + F_{12} \frac{\partial \phi_y}{\partial y} \]

\[ M_y = B_{12} \frac{\partial u}{\partial x} + B_{11} \frac{\partial v}{\partial y} - D_{12} \frac{\partial^2 w}{\partial x^2} - D_{11} \frac{\partial^2 w}{\partial y^2} + F_{12} \frac{\partial \phi_x}{\partial x} + F_{11} \frac{\partial \phi_y}{\partial y} \]

\[ M_{xy} = B_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - 2D_{66} \frac{\partial^2 w}{\partial x \partial y} + F_{66} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \]

\[ M_{x}^f = E_{11} \frac{\partial u}{\partial x} + E_{12} \frac{\partial v}{\partial y} - F_{11} \frac{\partial^2 w}{\partial x^2} - F_{12} \frac{\partial^2 w}{\partial y^2} + H_{11} \frac{\partial \phi_x}{\partial x} + H_{12} \frac{\partial \phi_y}{\partial y} \]

\[ M_{y}^f = E_{12} \frac{\partial u}{\partial x} + E_{11} \frac{\partial v}{\partial y} - F_{12} \frac{\partial^2 w}{\partial x^2} - F_{11} \frac{\partial^2 w}{\partial y^2} + H_{12} \frac{\partial \phi_x}{\partial x} + H_{11} \frac{\partial \phi_y}{\partial y} \]

\[ M_{xy}^f = E_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - 2F_{66} \frac{\partial^2 w}{\partial x \partial y} + H_{66} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \]

\[ Q_y^f = A_{44} \phi_y \]

\[ Q_x^f = A_{44} \phi_x \]

\[ Q_{ij} = A_{ij} \phi_i \]

\[ \begin{align*}
I_1 &= \int_{-h/2}^{h/2} \rho dz, \\
I_2 &= \int_{-h/2}^{h/2} \rho z dz, \\
I_3 &= \int_{-h/2}^{h/2} \rho z^2 dz \\
I_4 &= \int_{-h/2}^{h/2} \rho f(z) dz, \\
I_5 &= \int_{-h/2}^{h/2} \rho f(z) dz, \\
I_6 &= \int_{-h/2}^{h/2} \rho f^2(z) dz \\
A_{ij} &= \int_{-h/2}^{h/2} Q_{ij} dz \\
B_{ij} &= \int_{-h/2}^{h/2} Q_{ij} z dz \\
D_{ij} &= \int_{-h/2}^{h/2} Q_{ij} z^2 dz \\
E_{ij} &= \int_{-h/2}^{h/2} Q_{ij} f(z) dz \\
F_{ij} &= \int_{-h/2}^{h/2} Q_{ij} f(z) dz \\
H_{ij} &= \int_{-h/2}^{h/2} Q_{ij} f^2(z) dz \\
A_{44} &= \int_{-h/2}^{h/2} Q_{44} \left( \frac{df(z)}{dz} \right)^2 dz, \\
A_{55} &= A_{44} \\
Q_{11} &= \frac{E(z)}{1 - nu^2}, \\
Q_{12} &= \frac{nu E(z)}{1 - nu^2}, \\
Q_{44} &= Q_{55} = Q_{66} = \frac{E(z)}{2(1 + \nu)} \\
E(z) &= (E_c - E_m) \left( \frac{1}{2} + \frac{z}{h} \right)^p + E_m 
\end{align*} \]

where \( E_c \) and \( E_m \) denote the elasticity modulus of the ceramic and metal, respectively. \( p \) is power law index.
Discretization of the Governing Equations and Boundary Conditions

The simply supported boundary conditions and the governing equations are satisfied by the following displacement functions[6].

\[
\begin{align*}
  u &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \cos(\alpha x) \sin(\beta y) e^{i\omega t} \\
  v &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \sin(\alpha x) \cos(\beta y) e^{i\omega t} \\
  w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin(\alpha x) \sin(\beta y) e^{i\omega t} \\
  \phi_x &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} X_{mn} \cos(\alpha x) \sin(\beta y) e^{i\omega t} \\
  \phi_y &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn} \sin(\alpha x) \cos(\beta y) e^{i\omega t}
\end{align*}
\]  

(12)

where \( \alpha = \frac{m\pi}{a}, \beta = \frac{n\pi}{b} \), \( \omega \) is the natural circular frequency. Substituting Eq. (12) into Eq. (8), and collecting the coefficients, The following equation can be obtained:

\[
[K - \omega^2 M] \{\Delta\} = \{0\}, \{\Delta\}^T = \{U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn}\}
\]  

(13)

The natural circular frequency \( \omega \) can be obtained by solving the eigenvalue equations (13).

Numerical Examples

A square functionally graded plate of side a and thickness h with simply supported edges is considered. The functionally graded plate comprised of Metal (Aluminum, Al) and Ceramic (Zirconia, ZrO2). The material properties of Al and ZrO2 are as follows:

\[
\begin{align*}
  \text{Al:} & \quad E_m = 70 \text{GPa}, \quad v_m = 0.3, \quad \rho_m = 2702 \text{ kg/m}^3; \\
  \text{ZrO2:} & \quad E_c = 200 \text{GPa}, \quad v_c = 0.3, \quad \rho_c = 5700 \text{ kg/m}^3;
\end{align*}
\]

In this study, the variation of Young’s modulus E is given by Eq. (11). The Poisson’s ratio is assumed to be a constant through the thickness. The density of functionally graded material is given by Eq. (14)[7].

\[
\rho(z) = (\rho_c - \rho_m) \left( \frac{1}{2} + \frac{z}{h} \right)^p + \rho_m
\]

(14)

Natural frequencies are non-dimensionalized by

\[
\omega = \omega (a^2 / h) \sqrt{\rho_m / E_m}
\]

(15)

The first 3 natural frequencies for the fundamental vibration mode \( m=n=1 \) of a square simply supported functionally graded plate computed by various shear deformation theory are listed in Table 1-3 and compared with the three-dimensional exact solutions of Vel and Batra (2004) and the two-dimensional Navier solutions of Matsunaga (2008). According to the Table 1-3, the present results are in good agreement with those of Matsunaga (2008), and greater than those of Vel and Batra (2004).
Figs. 1 show the fundamental frequencies of a simply supported square functionally graded plate by the various theories \((a/h=5, 10, m=n=1)\). It can be seen from the Fig. 1 that results of Karama theory are greater than those of other theories. When the \(p=2\), the fundamental frequencies reach to the minimum value.

Figs. 2 show the fundamental frequencies of a simply supported square functionally graded plate by the various theories \((p=1, 10, m=n=1)\). According to the Figs. 2, results of Karama theory are greater than those of other theories. When the \(a/h>50\), the fundamental frequencies vary slightly with the increase of \(a/h\).

Table 1. First 3 natural frequencies of a simply supported square functionally graded plate, \(p=1, a/h=5, (m=n=1)\).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5.4806</td>
<td>5.7123</td>
<td>5.6929</td>
<td>5.7087</td>
<td>5.7994</td>
<td>5.6920</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. First 3 natural frequencies of a simply supported square functionally graded plate, \(p=1, a/h=10, (m=n=1)\).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>49.013</td>
<td>51.795</td>
<td>51.866</td>
<td>51.866</td>
<td>51.868</td>
<td>51.866</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. First 3 natural frequencies of a simply supported square functionally graded plate, \(p=3, a/h=5, (m=n=1)\).

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5.5285</td>
<td>5.6757</td>
<td>5.6564</td>
<td>5.6695</td>
<td>5.7897</td>
<td>5.6571</td>
<td></td>
</tr>
</tbody>
</table>

Figure 1. Fundamental frequencies of a simply supported square functionally graded plate by the various theories.
Conclusions

The various higher-order shear deformation theories are used to analyze the free vibration of functionally graded plate. A navier-type analytical method is used to solve the governing differential equations. Natural frequencies of simply supported functionally graded plates are calculated. According to the comparison of the present results of various theories with available published results, results of Karama theory are greater than those of other theories.

References


