Synchronization Method for a Class of Fractional-order and Integer-order Chaotic Systems

Lei-ping ZHU
Xuzhou University of Technology

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Abstract. This paper focuses on the synchronization issues between a class of fractional-order and integer-order chaotic systems. A closed-loop control system is introduced following the linear feedback control and fractional-order stability theories to address the synchronization issues. Appropriate coefficients in this paper mentioned synchronization are adopted to guarantee the finite time asymptotical stability of resulting synchronization error due to the disturbances. The proposed control scheme is validated using simulations, and the results illustrate that the proposed controller can implement the synchronization between a class of fractional-order chaotic systems and integer-order chaotic systems, two variable structure fractional-order chaotic systems or two mismatched fractional-order chaotic systems.

Introduction

As a special phenomenon in non-linear dynamical systems, chaos is investigated in scientific and engineering disciplines, for example, power electronic circuit (A. Harb, 2003), power systems (Harb and Nabi, 2003), medicine (Coffey, D. S., 1998), biology (Ditto W L, 1966), traffic problem (Cheng A, 2016), cardiac conduction model (M.A. Quiroz-Juárezab, 2016), wireless communication (Ren HP, 2016), combustor (Kabiraj L, 2015), power electronic inverter (Kabiraj, L., 2015) etc. Chaos has become one of the main focuses of research in non-linear systems. In 1990, Pecora and Carroll put forward the first solution for chaos synchronization problem. In the past few decades, scholars have proposed and realized a series of synchronization-methods: non-linear feedback synchronization (Jolly K. john, 1994), a control method based on the Kalman filter (Brandt, 1997), active control method (Agiza, 2001), variable structure synchronization via sliding mode (Hosseinnia, 2010), impulse control method for synchronization of a class of continuous systems (Wang, 2004), adaptive synchronization (Bernardo, 1996), different structures synchronization (Song, X., 2016), lag synchronization (Rosenblum, 1997) and projective synchronization in modified unified chaotic system (Liang, 2005).

Fractional-order chaotic systems are widely used in critical applications such as secure communication; hence, more attention is paid to the synchronization of chaotic systems. The existing research on synchronization mainly focuses on the integer-order chaotic systems or fractional-order systems (BehinFaraz, R., 2016; Singh, A. K., 2016; Li, R., 2014). The research on the synchronization between fractional-order and integer-order chaotic systems and the synchronization between fractional-order systems having different fractional orders are rarely reported. A general method is needed to a class of fractional-order systems.

On account of the tracking control theory the synchronization between a class of fractional-order chaotic systems and integer-order chaotic systems can be realized by tracking the output of an integer-order chaotic system (the reference signal) with the output of a fractional-order chaotic system. Then, the non-linear chaotic synchronization error system can be converted into a linear system through an appropriate feedback control method. As the controller is based on the stability theory of fractional linear systems, synchronization between the fractional-order and the integer-order chaotic systems can be guaranteed. It is also shown in the theoretical analysis and numerical simulation that the proposed method is suitable for fractional-order chaotic systems with different structures or different fractional-orders.
This paper is organized as follows: Section 2 introduces briefly the method of synchronization mentioned above. Section 3 illustrates three different situations of the synchronization method and an actual application in a coupling dynamos system. Finally, Section 4 concludes the paper.

Preliminaries

In this section, the system design is introduced with some of the important definitions, lemmas, and theorems. In this paper, it is assumed that the matrices presented will have compatible dimensions and is not mentioned explicitly.

**Definition 1** (KB Oldham, 1974; Podlubny, 1999; Aghababa, M. P., 2013).

According to the Riemann–Liouville derivative, the \(\alpha\)-th order derivative of a continuous function \(f: \mathbb{R}^+ \rightarrow \mathbb{R}\) is defined as follows

\[
D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{m-\alpha+1}} d\tau, \quad m-1 < \alpha < m
\]

\[
\frac{d^m}{dt^m} f(t), \quad \alpha = m
\]

where \(\Gamma\) is a Gamma function, and \(\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt\).

**Lemma 1** (Matignon, 1996; Wang, Z., 2013). As for an autonomous system \(D_t^\alpha x(t) = Ax(t)\), the necessary and sufficient condition for its asymptotic stability is given as follows

\[
\left|\arg(\text{spec}(A))\right| > \frac{\alpha\pi}{2}, \quad \alpha \in (0,1)
\]

Here, \(\text{spec}(A)\) means the eigenvalues of matrix \(A\) and \(\arg(.)\) expresses the argument function. The system will be stable if matrix \(A\) satisfies the condition mentioned in (2). Also, if \(\left|\arg(\text{spec}(A))\right| = \frac{\alpha\pi}{2}\), there exist multiple roots in eigenvalues of matrix \(A\).

**Theorem 1** Through the non-linear state space change of coordinates and nonlinear state feedback, the original non-linear system can be transformed into a controllable and observable system, where both the state equation and the output equation are linear. Also, the linear differential relation between the output \(y(t)\) and the input \(u(t)\) is established. Then, we can take advantage of the linear control theory to construct the controller.

The system is assumed as

\[
\dot{x} = f(x, u) \\
y = g(x)
\]

The final objective is to realize the output \(y(t)\) tracks after a desired trajectory \(y_d(t)\). Here, \(y_d(t)\) and its high enough derivative of time is known and bounded. The output \(y\) depends on \(u\) indirectly through the state variables and non-linear equation. Therefore, it is not easy to find an appropriate input \(u\) that can control the state of \(y\). Theorem 1 shows that through the feedback linearization a simple and a direct relationship between \(y\) and \(u\) can be obtained that simplify the design.

Generally speaking, feedback linearization is designed to realize the linearization of the system through nonlinear state feedback.

**Theorem 2** The tracking control of system means achieving the asymptotic tracking and suppressing the interference at the same time.

Consider system
\[ \dot{x} = Ax + Bu + B_w w, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \]
\[ y = Cx + Du + D_w w, \quad y \in \mathbb{R}^p, w \in \mathbb{R}^p \]

When the above system is under the influence of a reference signal and disturbance, if there is a corresponding control law to make the formula established:

\[ \lim_{t \to \infty} e(t) = \lim_{t \to \infty} (y_0(t) - y(t)) = 0 \]  
(5)

Then we call the system is astatic tracking.

Further, when the interference is inexistence for any reference signal, it should satisfy the following equation

\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} y_0(t) \]  
(6)

Also, when reference signal is inexistence for any interference, it should satisfy the equation below

\[ \lim_{t \to \infty} y(t) = 0 \]  
(7)

**System Design**

Consider the following driving system:

\[ D^\alpha_t x = Ax + f(x) \]  
(8)

Here, \( \alpha (0 < \alpha \leq 1) \) is differential order, if \( \alpha = 1 \), then the system (1) is an integer-order system. \( x(t) \in \mathbb{R}^n \) is a state variable of the system, \( A \in \mathbb{R}^{n \times n} \) is the linear part and \( f(x): \mathbb{R}^n \to \mathbb{R}^n \) is the non-linear part.

According to the drive system, a responding system can be described as follow:

\[ D^\beta_t y = By + g(y) \]  
(9)

Here, \( \beta (0 < \beta \leq 1) \) is differential order, if \( \beta = 1 \), then it is an integer-order system. \( y(t) \in \mathbb{R}^n \) is a state variable of the system, \( B \in \mathbb{R}^{n \times n} \) is the linear part and \( g(y): \mathbb{R}^n \to \mathbb{R}^n \) is the non-linear part.

According to the tracking control theory, the output signal of driving system given by (8) is considered as a reference signal. The control signal is added to the responding system such that the output signal of responding system given by (9) tracks the output signal of driving system. The responding system with the controller is given as:

\[ D^\beta_t y = By + g(y) + u(t) \]  
(10)

Here \( u(t) \) is the control term and is considered as:

\[ u(t) = u_1(t) + u_2(t) \]  
(11)

Since the differential orders of the driving system and the responding system are different, a compensating part is added to the controller to ensure the synchronization of the two systems. That is, driving system need the same fractional-orders in the responding system’s:

\[ u_1(t) = D^\beta_t x \]  
(12)

Then That is to say, driving system need the same fractional-orders in the responding system’s: system (10) can be rewritten as:

\[ D^\beta_t y = By + g(y) + D^\beta_t x + u_2(t) \]  
(13)

The synchronization error for the system is defined as
\[ e = y - x \quad (14) \]

To achieve the synchronization between the responding system (10) and the driving system (8), an appropriate control component \( u_2(x) \) is designed to satisfy:

\[ \lim_{t \to \infty} \|e\| = \lim_{t \to \infty} \|y - x\| = 0 \quad (15) \]

The control component \( u_2(t) \) is selected as:

\[ u_2(t) = -Bx - g(y) - Ke \quad (16) \]

Here \( K \in \mathbb{R}^n \), \( K \) is a variable. The formula of the total controller can be described as:

\[ u(t) = D^\alpha x - Bx - g(y) - Ke \quad (17) \]

Substituting (16) in (13),

\[ D^\alpha y = By + g(y) + D^\alpha x - Bx - g(y) - Ke \quad (18) \]

Equation (18) can be simplified as:

\[ D^\alpha e = (B - K)e \quad (19) \]

According to the stability theory of fractional linear system, if one can pick up an appropriate \( K \) to let \( |\arg(\lambda)| > \frac{\pi}{2} \) established then the zero point of the system (19) will be stable, which means \( \lim_{t \to \infty} e = 0 \). The above argument says that the synchronization between the system (8) and system (9) can be achieved.

**Conclusion**

In terms of stability theory of fractional linear system, combined with tracking control and feedback linearization, this paper proposes a synchronization controller design to achieve the synchronization between a class of fractional-order chaotic system and integer-order chaotic system. The synchronization between a fractional-order Lu chaotic system and an integer-order Liu chaotic system, different structures synchronization between fractional-order Liu and fractional-order Chen chaotic system and different fractional-orders Lorenz chaotic systems are studied in detail. The combination between the fractional-order chaotic system and the integer-order chaotic system, the fractional-order chaotic systems with different structures or different fractional-orders have made a great leap for further research and application of fractional-order chaotic system. The contribution of this paper is that it provided a simplified control method for controlling a complicated fractional-order chaotic systems.

**Reference**


