Isogeometric Analysis Method for Solving Parabolic PDEs by Using Bivariate Spline

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Abstract. In this paper, we study isogeometric analysis method for solving parabolic Pdes by using bivariate spline finite elements on domains defined by NURBS. We constructed bivariate spline proper subspace of $S^{2\times 3}(\Delta^{2\times n})$ which satisfies homogeneous boundary conditions on type-2 triangulations and quadratic B-spline interpolating boundary functions. A numerical test is solved to assess the accuracy of this method.

Introduction

Solving the parabolic pdes is much important and has recently been studied by several scholars. There are lots of literatures devoted to it. Researchers can referred to [1] for excellent surveys. We review some methods referred in this paper.

Recent works indicate that isogeometric analysis method have added yet another dimension to spline’s use, especially the simulation and numerical modelling [2-4]. Because of a strategy that can increase the regularity of functions through the mesh’s interfaces, and also to reduce the number of degrees of freedom, the isogeometric analysis method is undoubtedly the emergence [5]. More attempt show that the isogeometric analysis method which use geometric transformations and non uniform splines is much simpler for the other approaches [6]. Modern finite element techniques for parabolic pdes rely on ideas from differential geometry and more precisely on the existence of discrete spaces [7]. Adaptive mesh refinement with spline method is not as straightforward which ideas have been proposed in [8]. This problem arises in more and more branches of science. In particular, plasma physics, heat conduction process, electrochemistry, semi-conductor modeling, control theory, inverse problems and biotechnology [9]. The analysis, development and implementation of numerical methods for the solution of such parabolic pdes has received wide attention in the literature [10].

The organization of this paper is as follows. In Section 2, we construct the bivariate spline space. We write an solution to the general parabolic peds in a weak form in Section 3. Also in Section 3, we propose isogeometric analysis method for solving the weak solution. In Section 4, a numerical examples are given, also included in Section 4 is the comparison of the Isogeometric analysis method and other methods. A conclusion is drawn in Section 5.

Bivariate Spline Space

Splines are polynomials which are piecewise and have certain smoothness. The space of bivariate splines with degree $k$ and smoothness $\mu$ over $\Delta$ is defined by

$$S^\mu_k(\Delta) := \{ s \in C^\mu(\Omega) \mid s|_{\delta_i} \in P_k, i = 1, \ldots, M \}$$

Here, the partition $\Delta$ of the domain $\Omega$ is carried out by using a finite number of irreducible algebraic, and $\delta_1, \ldots, \delta_M$, indicate cells of $\Delta$. We call type-2 triangulations are uniform type-2 triangulations if the original rectangular partition is uniform, see Figure. 1.
Now, we would like to construct the locally supported splines in \( S_{4}^{2}(\Delta_{mn}^{(2)}) \) which are consist of three classes of \( C^2 \) quartic B-spline bases. Also, \( S_{4}^{2,3}(\Delta_{mn}^{(2)}) \) can be constructed by adding the following two continuous conditions: (i) \( s \) is \( C^2 \) continuous on the rectangle grid segments; (ii) \( s \) is \( C^1 \) continuous on the diagonal grid segments. Next, we discuss locally supported splines in proper subspace of \( S_{4}^{2,3}(\Delta_{mn}^{(2)}) \) with homogenous boundary conditions on type-2 triangulations. The basic idea is to use the linear combination of \( B(x, y) \) in \( S_{4}^{2,3}(\Delta_{mn}^{(2)}) \) and their translations.

Let

\[
B_{ij}(x, y) = B(mx - i, ny - j).
\]

Define the basis functions \( \tilde{B}_{i,y}(x, y) \) as follows:

\[
\begin{align*}
\tilde{B}_{1,1}(x, y) &= B_{1,1}(x, y) - B_{1,-1}(x, y), \\
\tilde{B}_{i,1}(x, y) &= B_{i,1}(x, y) - B_{i,-1}(x, y) - B_{1,1}(x, y) + B_{1,-1}(x, y), \\
\tilde{B}_{i,2}(x, y) &= B_{i,2}(x, y) - B_{i,-2}(x, y) - B_{1,1}(x, y) + B_{1,-1}(x, y), \\
\tilde{B}_{i,i}(x, y) &= B_{i,i}(x, y) - B_{i,-i}(x, y) - B_{1,1}(x, y) + B_{1,-1}(x, y) \quad (i = 2, 3, \ldots, m-2) \\
\tilde{B}_{m,1}(x, y) &= B_{m,1}(x, y) - B_{m,1}(x, y), \\
\tilde{B}_{m,i}(x, y) &= B_{m,i}(x, y) - B_{m,-i}(x, y) - B_{1,1}(x, y) + B_{1,-1}(x, y) \quad (i = 2, 3, \ldots, m-2) \\
\tilde{B}_{i,n}(x, y) &= B_{i,n}(x, y) - B_{i,-n}(x, y), \\
\tilde{B}_{i,i}(x, y) &= B_{i,i}(x, y) - B_{i,-i}(x, y) - B_{1,1}(x, y) + B_{1,-1}(x, y) \quad (i = 2, 3, \ldots, n-2) \\
\tilde{B}_{2,2}(x, y) &= B_{2,2}(x, y) - B_{2,-2}(x, y), \\
\tilde{B}_{j,j}(x, y) &= B_{j,j}(x, y) - B_{j,-j}(x, y) - B_{2,2}(x, y) + B_{2,-2}(x, y) \quad (j = 2, 3, \ldots, n-2)
\end{align*}
\]

B-spline functions in Equation.(1)-(3) are called corner, side and interior B-spline bases, respectively. Their supports are shown in Figure 2. The B-spline functions are \( C^1 \) across the single marked mesh lines and \( C^0 \) across the double marked mesh segments.

It can be proved that \( \tilde{B}_{ij}(x, y) : 1 \leq i \leq m-1, 1 \leq j \leq n-1 \) can only span the proper subspace of \( S_{4}^{2,3}(\Delta_{mn}^{(2)}) \) with homogenous boundary conditions on type-2 triangulations (\( S_{4}^{2,3,0}(\Delta_{mn}^{(2)}) \) for short).

Isogeometric Analysis Method for Solving Parabolic PDEs

We focus on the following problem: Find \( u = u(x, y) \) such that
\( u_y - u_{xx} = f, \quad \text{in} \Omega, \quad t > 0 \) \tag{4}

\( u = 0, \quad \text{on} \partial \Omega, \quad t > 0 \) \tag{5}

\( u(\cdot, 0) = u_0, \quad \text{in} \Omega \) \tag{6}

Eq. (4)-Eq. (6) can be given the following equivalent weak formulation: Find \( u : R \rightarrow H^1_0(\Omega) \) such that

\[ (u_t, v) + (\nabla u, \nabla v) = (f, v), \quad \forall v \in H^1_0(\Omega), \quad t > 0 \]

\[ u(\cdot, 0) = u_0. \]

The Isogeometric Analysis Method is defined as follows: Let \( 0 = t_0 < t_1 < \cdots < t_j < \cdots \) be a (not necessarily) partition of the positive \( t \)-axis \( R \) into subintervals \( I_j = (t_{j-1}, t_j] \), and define with \( q \) a nonnegative integer the corresponding set of piecewise polynomials of degree at most \( q \) in \( t \) with values \( \text{in} \ H^1_0(\Omega) \) by

\[ W = \{ v : \left. v \right|_{t_i} = \sum_{i=0}^{q} a_i f^i, a_{i,j} \in H^1_0(\Omega), i = 0, 1, \ldots, q, l = 1, 2, \ldots \} . \]

In this note we shall consider only the case \( q = 0 \), it means that the solution of problem Eq. (4)-Eq. (6) in each subintervals \( I_j \) (for some suitable) is not changed.

\[ (U_i - U_{i-1}, v) + k_i (\nabla U_i, \nabla v) = \int_{t_i} (f, v) dt, \quad \forall v \in H^1_0(\Omega), \quad l = 1, 2, \ldots, \]

where \( k_i = t_i - t_{i-1} \) is the length of the subinterval \( I_j \). \( U_i \) is the numerical solution of Eq. (4)-Eq. (6) when \( t \in I_i \).

Since \( S^2_{4,3,0}(\Delta^{(2)}_{mn}) \) can be embedded into \( H^1_0(\Omega) \), we can select it as the testing function space.

To find a solution \( U_i \in S^2_{4,3,0}(\Delta^{(2)}_{mn}) \) such that

\[ (U_i - U_{i-1}, v) + k_i (\nabla U_i, \nabla v) = \int_{t_i} (f, v) dt, \quad \forall v \in S^2_{4,3,0}(\Delta^{(2)}_{mn}) \].

It is equivalent to the following formula:

\[ (U_i - U_{i-1}, \tilde{B}, v) + k_i (\nabla U_i, \nabla \tilde{B}, v) = \int_{t_i} (f, \tilde{B}, v) dt, \quad \forall \tilde{B}, v \in S^2_{4,3,0}(\Delta^{(2)}_{mn}) \]. \tag{7}

By using the B-spline bases on \( S^2_{4,3,0}(\Delta^{(2)}_{mn}) \), we can write

\[ U_i = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \tilde{\lambda}_{i,j} \tilde{B}_{i,j}(x, y). \]

and insert to Eq. (7), we have the following linear system

\[ \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \tilde{\lambda}_{i,j} (\tilde{B}_{i,j}, \tilde{B}, v) + k_i \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \tilde{\lambda}_{i,j,n} (\nabla \tilde{B}_{i,j}, \nabla \tilde{B}, v) = \int_{t_i} (f, \tilde{B}, v) dt + (U_i, \tilde{B}, v), \quad \forall 1 \leq s \leq m - 1, 1 \leq t \leq n - 1 \]

Therefore, the coefficients \( \tilde{\lambda}_{i,j} \) can be determined by the system of linear equations Eq. (8).

**Numerical Test**

In this section, an example is provided to illustrate the proposed isogeometric analysis method to solve a parabolic pde.
Let $\Omega = (0, 1) \otimes (0, 1)$, consider the linear parabolic equation
\[ u_y - u_{xx} = 0, \quad 0 \leq x \leq 1, \quad y > 0, \]
with the following initial and boundary conditions:
\[ u(x, 0) = 1, \quad 0 \leq x \leq 1, \]
\[ u(0, y) = u(1, y), \quad y > 0. \]

Here, we choose $m$ and $n$ are 32 in spline space $S_{i, 3}^{2, 0}(\Delta_{mn})$. The exact solution $u(x, y)$, the approximate solution $\hat{u}(x, y)$ by using the isogeometric analysis method, the approximate solution $\tilde{u}(x, y)$ by the finite difference method and the approximate solution $\tilde{u}(x, y)$ by using RBF method proposed are shown in Figure 3 and Figure 4.

The comparison of the numerical solutions by using the isogeometric analysis method, the RBF method, the finite difference method (FD method) and the finite element method (FE method) with its exact solutions at $t=0.02$, $0.04$, $0.06$, $0.08$, $0.1$ are displayed in Table 1, where $x \in (0, 1)$. Here, we enumerate the 2-norm errors between the exact solutions and the numerical solutions obtained from some methods. From Table 1, we can see that the GBC method and RBF method have fine accuracy. However, we have to choose 33 points to calculate the numerical solutions if we use the RBF method, it says, we should solve a system with the size of $33 \times 33$.

<table>
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<tr>
<th></th>
<th>FD method</th>
<th>FE method</th>
<th>RBF method</th>
<th>isogeometric analysis method</th>
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<tr>
<td>$t=0.02$</td>
<td>4.327609e-002</td>
<td>6.163390e-003</td>
<td>4.321798e-003</td>
<td>3.235934e-004</td>
</tr>
<tr>
<td>$t=0.04$</td>
<td>6.867344e-003</td>
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<td>8.137646e-004</td>
<td>6.673442e-004</td>
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<td>$t=0.06$</td>
<td>4.426754e-003</td>
<td>8.416654e-004</td>
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<td>$t=0.08$</td>
<td>4.890322e-003</td>
<td>7.344521e-004</td>
<td>6.435536e-004</td>
<td>5.704321e-004</td>
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<tr>
<td>$t=0.1$</td>
<td>3.742426e-004</td>
<td>3.652153e-004</td>
<td>8.164187e-005</td>
<td>6.357666e-005</td>
</tr>
</tbody>
</table>

![Figure 3](a) (a) $u(x, y)$ (b) $\hat{u}(x, y)$

![Figure 4](c) (c) $\tilde{u}(x, y)$ (d) $\tilde{u}(x, y)$

362
Summary
In this paper, an isogeometric analysis method based on bivariate spline has been proposed to solve the general parabolic pdes. Here, bivariate spline proper subspace of $S^2_2(\Delta^{(2)}_{m,n})$ satisfying homogeneous boundary conditions on type-2 triangulations and quadratic B-spline interpolating boundary functions are primarily constructed. We could get the numerical solutions of parabolic pdes by using the spline in $S^2_2(\Delta^{(2)}_{m,n})$. The feasibility of the method is shown by a numerical test and the approximated solutions are found to be in good agreement with the known exact solutions.

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References