Study on Fuzzy Quadratic Programming Problems

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Abstract. Under the condition of D-differentiable, fuzzy quadratic programming, whose objective functions are fuzzy quadratic and constraint conditions are fuzzy linear, are studied. First, we prove that fuzzy quadratic programming are a special case of convex fuzzy programming, whose objective mappings and constraint conditions are both D-differentiable; Then by the KKT conditions of convex fuzzy programming, we propose the optimality KKT conditions of fuzzy quadratic programming.

Introduction

Fuzzy mathematics decision methods were originally put forward by Bellman and Zadeh\cite{1} in 1970. Since then, with the efforts of many learned men, substantial achievements in the domain of fuzzy programming have been yielded\cite{2,3,4,5,6}. In particular, fuzzy programming, whose objective functions are fuzzy quadratic and constraint conditions are fuzzy linear, are called fuzzy quadratic programming\cite{6}.

Being more practical in dealing vagueness and uncertainty, fuzzy quadratic programming have aroused great attention from scholars, and it has yielded a series of good results\cite{7,8}. It’s commonly known that Kuhn-Tucker conditions play a pivotal role in the classical solution of mathematical programming. Zhang et al.\cite{8} introduced the concept of sub-differential and differential from the perspective of convex analysis in 2005. In Ref.\cite{8}, saddle-point and minimax theorem in vague sense were discussed and applied to the Lagrange duality theory of fuzzy programming, and Lagrange duality and KKT conditions of convex fuzzy programming were obtained; meanwhile, fuzzy quadratic programming were studied, besides, corresponding Lagrange duality and KKT conditions were proposed.

In Refs.\cite{9,10}, Bao et al. developed the D-differentiable of generalized fuzzy mapping and studied the convex fuzzy programming and its Lagrange duality problem, as well, a more reasonable and concise form of the KKT conditions of convex fuzzy programming had been derived. In this paper, based on D-differentiable, fuzzy quadratic programming will be further discussed; KKT conditions for general case will be established besides.

Preliminaries

Definition 2.1\cite{8,9}. Let $R$ denote the set of all real numbers. A fuzzy number $[0,1]$ will be a fuzzy set $u: R \to [0,1]$ with the following properties (i)-(iv):

(i) $u$ is normal, i.e., there exists an $x_0 \in R$, such that $u(x_0) = 1$;

(ii) $u$ is convex, i.e.,

$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ whenever $x, y \in R$ and $\lambda \in [0,1]$;

(iii) $u$ is upper semi-continuous;

(iv) $cl(\text{supp } u) = cl\{x \in R | u(x) > 0\}$ is a compact set.
The family of all fuzzy numbers will be denoted by $\mathcal{F}$. For $\alpha \in [0,1]$, the $\alpha$-level sets of a fuzzy number $u \in E$ is a closed interval $[u]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$. We call $u = ([u(\alpha), \overline{u}(\alpha), \alpha]_{0 \leq \alpha \leq 1})$ is the parametric expression of fuzzy number $u \in E$.

For $u, v \in \mathcal{F}$ and $r \in \mathbb{R}$, the addition and scalar multiplication on $F$ can be represented as:

$$u + v = ([u(\alpha) + v(\alpha), \overline{u}(\alpha) + \overline{v}(\alpha), \alpha]_{0 \leq \alpha \leq 1}), ru = ([ru(\alpha), r\overline{u}(\alpha), \alpha]_{0 \leq \alpha \leq 1}),$$

where

$$u + v = u(\alpha) + v(\alpha), \overline{(u + v)}(\alpha) = \overline{u}(\alpha) + \overline{v}(\alpha), ru = ru(\alpha), \overline{ru} = r\overline{u}(\alpha).$$

For $u \in \mathcal{F}$ and $a \in \mathbb{R}$, we can easily obtain $u - a \in \mathcal{F}$,

$$u - a = u(\alpha) - a \quad \text{and} \quad \overline{u - a}(\alpha) = \overline{u}(\alpha) - a \quad \text{for any} \quad \alpha \in [0,1].$$

Let $F : M \rightarrow \mathcal{F}$, $G_i : M \rightarrow \mathcal{F}$ $(i = 1, 2, \ldots, n)$ are all convex fuzzy mappings, then we call

$$(FCP) \quad \left\{ \begin{array}{l} \min F(x) \\ s.t. G_i(x) \leq 0, (i = 1, 2, \ldots, n) \end{array} \right., \quad x \in M,$$

is the convex fuzzy programming.

**Definition 2.2**. Let $A = \{\alpha \rightarrow (\underline{a}_\alpha(\alpha), \overline{a}_\alpha(\alpha)) | 0 \leq \alpha \leq 1\}$ be a fuzzy matrix. For any $\alpha \in [0,1]$, $\underline{A}(\alpha) = (\underline{a}_\alpha(\alpha))_{\alpha=0}^1$ and $\overline{A}(\alpha) = (\overline{a}_\alpha(\alpha))_{\alpha=0}^1$ are $n \times n$ symmetric positive definite matrices, then $A$ is said to be symmetric positive definite.

**Definition 2.3**. Let $F : M \rightarrow \mathcal{F}$ be a fuzzy mapping, $x^0 = (x_{11}^0, x_{12}^0, \ldots, x_{1n}^0) \in \text{int} M$. If there exists $u = (u_1, u_2, \ldots, u_n) \in \mathcal{F}^n$ such that

$$\lim_{x \rightarrow x^0} \frac{F(x) + \left\{ u, (x - x^0) \right\}^\top F(x^0) + \left\{ u, (x - x^0) \right\}^\top}{d(x, x^0)} = 0,$$

then $F$ is said to be D-differential at $x^0$, and $\nabla F(x^0) = (u_1, u_2, \ldots, u_n)$ is referred to as the gradient of $F$ at $x^0$.

In [10], the following conclusions about convex fuzzy programming had been gotten.

**Theorem 2.1**. Let $X = \{x | G_j(x) \leq 0 (j = 1, 2, \ldots, m)\}, x^0 \in X$ and $F, G_j (j = 1, 2, \ldots, m)$ is D-differential $x^0$. If there exists $\lambda^0 = (\lambda_{11}^0, \lambda_{12}^0, \ldots, \lambda_{1m}^0) \in \mathbb{R}^n$, such that

$$\left\{ \begin{array}{l} \nabla F(x^0) + \sum_{j=1}^m \lambda_{ij}^0 \nabla G_j(x^0) = 0 \\ \sum_{j=1}^m \lambda_{ij}^0 G_j(x^0) = 0 \end{array} \right., \quad (i = 1, 2, \ldots, m)$$

Then $(x^0, \lambda^0) \in M \times \mathbb{R}^m$ is a saddle-point of $L(x, \lambda)$. Conversely, $(x^0, \lambda^0)$ is a saddle-point of $L(x, \lambda)$, then $x^0$ is optional solution to problem (FCP) and $(x^0, \lambda^0)$ satisfies the condition (1).

**Theorem 2.2**. Let $F, G$ be fuzzy mappings $(M \rightarrow \mathcal{F})$, and be D-differentiable at point $x^0 \in \text{int} M$, then $F + G$ is D-differentiable at point $x^0$ and $\nabla (F + G)(x^0) = \nabla F(x^0) + \nabla G(x^0)$.

**Theorem 2.3**. A point $(x^0, \lambda^0) \in M \times \mathbb{R}^m$ is a saddle-point of the $L(x, \lambda)$ if and only if $x^0$ and $\lambda^0$ are the optimal solutions of primal problem (FP) and dual problem (DFP) respectively, and their values are equal.
Main Results

In this section, a class of fuzzy quadratic programming problems will be discussed under the condition of D-differentiable. Next, let’s introduce the basic concepts about fuzzy quadratic programming problems[8].

Let \( n \times n \) fuzzy matrix \( Q \) is symmetric positive definite, \( b \in \mathbb{R}^m \), \( \tilde{b} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_m) \), \( C \) and \( A \) are \( n \)-dimensional fuzzy vector and \( m \times n \) fuzzy matrix, respectively. Then fuzzy quadratic programming problems

\[
\text{(FQP)} \quad \begin{cases}
\min C^T x + \frac{1}{2} x^T Q x \\
\text{s.t.} Ax \leq \tilde{b}, x \in \mathbb{R}^n
\end{cases}
\]

will be discussed in this section.

Let \( G(x) = Ax - \tilde{b} \), then by[8], we can easily obtain \( F(x) \) is convex fuzzy mapping.

Let \( G(x) = (G_1(x), \ldots, G_m(x)) \), \( A_j = (x_{j1}, x_{j2}, \ldots, x_{jn})^T \), then \( G_j(x) = A_j x - \tilde{b}_j (j = 1, 2, \ldots, m) \), (FQP) can be rewritten as follows

\[
\text{(FQP)} \quad \begin{cases}
\min F(x) \\
\text{s.t.} G_j(x) \leq 0 (j = 1, 2, \ldots, m)
\end{cases}
\]

and the corresponding Lagrangian of (FQP) is

\[
L(x, \lambda) = C^T x + \frac{1}{2} x^T Q x + \lambda^T (Ax - \tilde{b}), \quad \text{where} \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) (\lambda_j > 0, j = 1, 2, \ldots, m).
\]

And we can easily obtain \( L(x, \lambda) \) is a convex fuzzy mapping: \( \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathcal{F} \).

The Lagrange duality problem of (FQP) denoted by

\[
\text{(DFOP)} \quad \max_{\lambda \geq 0} d(\lambda), \quad \text{where} \quad d(\lambda) = \min_{x \in \mathbb{R}^n} L(x, \lambda).
\]

Proposition 3.1. Let \( n \times n \) fuzzy matrix \( Q \) is symmetric positive definite, then the fuzzy mapping \( h(x) = x^T Q x \) is D-differentiable at \( x^0 \in \text{int} \, \mathbb{R}^n \).

Proof. For any \( r \in [0, 1] \), we have \( h(x)(r) = x^T Q(r)x, \overline{h(x)}(r) = x^T \overline{Q}(r)x \), and for \( i = 1, 2, \ldots, n \),

\[
\frac{\partial}{\partial x_i} h(x)(r) = 2(a_i(r)x_i + a_{i2}(r)x_2 + \cdots + a_{in}(r)x_n)
\]

\[
\frac{\partial}{\partial x_i} \overline{h(x)}(r) = 2(\overline{a}_i(r)x_i + \overline{a}_{i2}(r)x_2 + \cdots + \overline{a}_{in}(r)x_n)
\]

are continuous at \( x^0 = (x^0_1, x^0_2, \ldots, x^0_n) \). Therefore real-valued functions \( \underline{h}(x)(r) \) and \( \overline{h}(x)(r) \) are differentiable at \( x^0 \). Hence, by the concept of differentiability from real analysis, we can obtain

\[
\underline{h}(x)(r) - \underline{h}(x^0)(r) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \underline{h}(x^0)(r)(x_i - x^0_i) + o(d(x, x^0)) \tag{2}
\]

\[
\overline{h}(x)(r) - \overline{h}(x^0)(r) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \overline{h}(x^0)(r)(x_i - x^0_i) + o(d(x, x^0)) \tag{3}
\]

On the other hand, by \( x_i \geq 0 (i = 1, 2, \ldots, n) \), for \( r \in [0, 1] \) we have
\[ \frac{\partial}{\partial x_i} h(x)(r) = \left[ 2(a_{nx_i}^0 + a_{2x_i}^0 + \cdots + a_{mx_i}^0) \right](r), \quad \frac{\partial}{\partial x_i} h(x)(r) = \left[ 2(a_{nx_i}^0 + a_{2x_i}^0 + \cdots + a_{mx_i}^0) \right](r), \]
\[ 2(a_{nx_i}^0 + a_{2x_i}^0 + \cdots + a_{mx_i}^0) \in \mathcal{F} \ (i = 1, 2, \ldots, n). \]

Take \( u_i = 2(a_{nx_i}^0 + a_{2x_i}^0 + \cdots + a_{mx_i}^0) \), then \( u_i \in \mathcal{F} \ (i = 1, 2, \ldots, n) \) and for any \( r \in [0, 1] \), we get
\[ u_i(r) = \frac{\partial}{\partial x_i} h(x)(r), u_i(r) = \frac{\partial}{\partial x_i} h(x)(r) \ (i = 1, 2, \ldots, n). \]

Therefore, for \( u = (u_1, u_2, \ldots, u_n) \), by (2) and (3) we have
\[ \lim_{x \to x^0} \frac{D\left(h(x) + \left(u,(x-x^0)\right), h(x^0) + \left(u,(x-x^0)\right)^T\right)}{d(x,x^0)} = \lim_{x \to x^0} \sup_{x \in [0,1]} \left[ \left\| \frac{h(x)(r) - h(x^0)(r)}{d(x,x^0)} - \sum_{i=1}^{n} u_i(x_i - x^0) \right\| \right] = 0. \]

So by definition 2.3, the fuzzy mapping \( h(x) \) is D-differentiable at \( x^0 \) and \( \nabla h(x^0) = 2Qx^0 \).

**Proposition 3.2.** Let \( A \) be \( m \times n \) fuzzy matrix, \( x = (x_1, x_2, \ldots, x_n) \in R^n \), \( b_j \in R \ (j = 1, 2, \ldots, m) \), then fuzzy mapping \( G_j(x) = A_J(x - b_j) \ (i = 1, 2, \ldots, m) \) is D-differentiable at \( x^0 \in \text{int } R^n \).

**Proof.** For \( \alpha \in [0,1] \), we have \( G_j(x)(\alpha) = A_J(\alpha)x - b_j \), \( \overline{G}_j(x)(\alpha) = \overline{A}_J(\alpha)x - b_j \), and for \( j = 1, 2, \ldots, m \), we can also obtain
\[ \frac{\partial}{\partial x_i} G_j(x)(\alpha) = a_{ji}(\alpha), \quad \frac{\partial}{\partial x_i} \overline{G}_j(x)(\alpha) = \overline{a}_{ji}(\alpha), \]
are continuous at \( x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \). Hence real-valued functions \( G_j(x)(\alpha) \) and \( \overline{G}_j(x)(\alpha) \) are differentiable at \( x^0 \).

By the concept of differentiability from real analysis, we get
\[ G_j(x)(\alpha) - G_j(x^0)(\alpha) = \sum_{i=1}^{n} a_{ji}(\alpha)(x_i - x_i^0) + o(d(x,x^0)), \quad (4) \]
\[ \overline{G}_j(x)(\alpha) - \overline{G}_j(x^0)(\alpha) = \sum_{i=1}^{n} \overline{a}_{ji}(\alpha)(x_i - x_i^0) + o(d(x,x^0)). \quad (5) \]

Take \( u_i = a_{ji} \), then \( u_i \in \mathcal{F} \ (i = 1, 2, \ldots, n) \) and for \( \alpha \in [0,1] \) we have \( u_i(\alpha) = a_{ji}(\alpha), \overline{u}_i(\alpha) = \overline{a}_{ji}(\alpha) \).

Hence, for \( u = (u_1, u_2, \ldots, u_n) \), by (4) and (5) we obtain
\[ \lim_{x \to x^0} \frac{D\left(G_j(x) + \left(u,(x-x^0)\right), G_j(x^0) + \left(u,(x-x^0)\right)^T\right)}{d(x,x^0)} = 0. \]

So, by definition 2.3, it’s easy to know \( G_j(x) \) is D-differentiable at \( x^0 \) and \( \nabla G_j(x) = (a_{j1}, a_{j2}, \ldots, a_{jm}) \).

**Proposition 3.3.** Let \( C \) be an n-dimensional fuzzy vector, then \( H(x) = C^T x \), \( x \in R^n \), is D-differentiable at \( x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in \text{int } R^n \) and \( \nabla H(x) = C^T \).

**Proof.** Similar to proposition 3.1 and proposition 3.1, it can be derived easily.
**Theorem 3.1.** In fuzzy quadratic programming (FQP), let 
\[ X = \{ x \in \mathbb{R}^n \mid Ax \leq \bar{b} \} , \quad x^0 \in X \] 
If there exits \( \lambda^0 = (\lambda_1^0, \lambda_2^0, \ldots, \lambda_m^0) \in \mathbb{R}^m \), such that
\[
\begin{align*}
C^T(r) + Q(r)x^0 + A^T(r)\lambda^0 &= 0 \\
C^T(r) + \bar{Q}(r)x^0 + A^T(r)\lambda^0 &= 0 \\
\lambda^0(A(r)x^0 - \bar{b}) &= \lambda^0(A(r)x^0 - \bar{b}) = 0,
\end{align*}
\]
then \((x^0, \lambda^0) \in R^n \times R^m\) is a saddle-point of \( L(x, \lambda) \); conversely, \((x^0, \lambda^0) \in R^n \times R^m\) is a saddle-point of \( L(x, \lambda) \), then \( x^0 \) is an optional solution of problem (FQP) and \((x^0, \lambda^0)\) satisfies the conditions (6).

**Proof.** by proposition 3.1 and proposition 3.3, we derive \( F(x) = C^T x + \frac{1}{2} x^T Q x \) is D-differentiable at \( x^0 \in X \) and \( \nabla F(x) = C^T + Q x^0 \). And by proposition 3.2, \( G_j(x) = A_j x - \bar{b}_j \) is D-differentiable at \( x^0 \) and \( \nabla G_j(x) = (a_{j1}, a_{j2}, \ldots, a_{jm}) \). Besides, (FQP) is a special case of convex fuzzy programming. So, according to theorem 2.1, this proof is completed.

**Remark 3.1.** The conditions (5) in theorem 3.1 are usually called KKT conditions for fuzzy quadratic programming (FQP).

**Remark 3.2.** By theorem 2.3, \( x^0 \) and \( \lambda^0 \) satisfying KKT conditions are the optional solutions of fuzzy quadratic programming (FQP) and dual fuzzy quadratic programming (DFQP), respectively, besides, they have equal optimal values.

**Remark 3.3.** Under more general differentiable conditions, theorem 3.1 is the generalization of the corresponding conclusions in [8].

**Summary**

Fuzzy quadratic programming (FQP) is a special kind of fuzzy programming problem. In this paper, based on D-differentiable, fuzzy quadratic programming is discussed. A more general KKT condition for fuzzy quadratic programming is given. For the further study of fuzzy quadratic programming problem has laid a good foundation.

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**References**


