Quantization of Three-junction Superconducting Flux Quantum Bit System

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Abstract. This work is aimed to present a method for the quantization of a many superconducting circuits coupling system. On the basis of rf-SQUID qubit, three Josephson junctions are connected in series in the superconducting loop to reduce the impact of the external electromagnetic environment on the bit. When there is an external magnetic flux in the superconducting ring, the Hamiltonian of the three-junction flux quantum bit is achieved. Firstly, ignoring the energy term caused by the applied potential in the kinetic energy term. Then the Lagrangian of the three-junction flux quantum bit is obtained by the Josephson junction characterization kinetic energy term. Secondly, canonical momentum is introduced to get the relationship of Hamiltonian potential energy. Finally, the potential energy is divided into two parts: the inherent potential energy of the system and the interaction energy of the system. Then, the Hamiltonian is discussed separately with additional time-dependent flux and no external flux.

Introduction

In recent years, with the rapid development of quantum communication and quantum computing, how to compute quantum in a real physical system has drawn much attention. For its ease of large-scale integration and scalability, solid quantum device has been considered as one of the alternatives which are most likely to take the lead in computing quantum and among those devices, Josephson structure super conductance quantum bits is the one with highest potential and fastest growing. Josephson junction can be used to design a specific superconducting circuit and to implement superconducting qubit. Currently, mainly illustrates three superconducting qubits that are widely used: superconducting phase qubits, superconducting flux qubit and superconducting charge qubits. The simplest of these is the rf-SQUID qubit, which is a loop of a Josephson junction and superconducting material[2].

In this paper, three Josephson junctions are connected in series and constructs into a small superconducting loop to reduce the influence of the external electromagnetic environment on the qubit. Then, a multi-superconducting circuits coupling system is quantized into a formation of two qubit Rabi models. The quantized three-junction flux quantum bit is useful for studying its own quantum properties. Those theoretical achievements are of great significance for people to utilize three-junction flux qubit and prolong its decoherence time.

Hamiltonian for Three-junction Flux Quantum Bits

By mean of rf-SQUID qubit, J. E. Mooij et al. connected the three Josephson structures in a smaller superconducting loop to reduce the influence of the external electromagnetic environment on the qubit[3]. The three Josephson junctions are connected in series in the superconducting loop, and the applied magnetic flux is $\Phi_{ex}$ in the superconducting ring. The equivalent circuit diagram of the three-junction magnetic flux qubit is displayed in Figure 1[4]. While the three-junction magnetic flux
qubit system satisfied \( C_1 = C_2 = C, E_{j_1} = E_{j_2} = E_j, \ C_3 = \alpha C \) and \( E_{j_3} = \alpha E_j \). The energy of the three-junction flux quantum bit system has potential energy and kinetic energy composition. Based on research works, we studied how to realize the multi-qubit quantum Rabi model.

![Figure 1. Three-junction flux quantum bit equivalent circuit diagram.](image)

The potential term of the three-junction flux quantum bit is the system of Josephson energy\(^2\)

\[
U = \sum_{i=1}^{3} E_{j_i} (1 - \cos \phi_i)
\]  

(1)

The energy term that caused by the external magnetic field of the loop in the three-junction flux quantum kinetic energy term has been ignored. In this case, the Lagrangian kinetic energy term based on the properties of the Josephson junction can be written as

\[
T = \frac{1}{2} (\Phi_0)^2 \cdot T \cdot \left( \Phi \cdot (\Phi_0 \cdot Q_0) \right)^T. 
\]  

(2)

Then, the Lagrangian of the three-junction flux qubit becomes

\[
L = \frac{1}{2} V^T CV - \frac{1}{2} V^T C V - \sum_{i=1}^{3} E_{j_i} (1 - \cos \phi_i)
\]  

(3)

Based on research works, introduce the canonical momentum \( P = (\Phi_0 / 2\pi) \cdot \frac{\Phi}{2\pi} Q_0 \), the system’s Hamiltonian can be simplified as\(^5\)

\[
H = \frac{1}{2M_p} \left[ P_p + \frac{\Phi_0}{2\pi} Q_p \right]^2 + \frac{1}{2M_m} \left[ P_m + \frac{\Phi_0}{2\pi} Q_m \right]^2 + U
\]  

(4)

where \( M_p = 2(\Phi_0 / 2\pi)^2 (1 + \gamma)C \), \( M_m = 2(\Phi_0 / 2\pi)^2 (1 + 2\alpha + \gamma)C \).

**Potential Energy of Hamiltonian Rabi Model**

Assuming that the external magnetic flux in the superconducting ring is divided into a constant p-art and a time-dependent part, then we can define

\[
\varphi_p = \frac{\varphi_1 - \varphi_2}{2},
\]

\[
\varphi_q = \frac{\varphi_1 + \varphi_2}{2} - \frac{\alpha}{1 + 2\alpha} \varphi_0(t).
\]
The potential energy term includes the potential energy of the system and the system interacts with the addition of time-dependent flux interactions \[3\]. We can get the potential energy of the system as

\[
U = E_j \left[ 2 + \alpha - 2 \cos \varphi_p \cos \varphi_q + \frac{\alpha}{1 + 2\alpha} \varphi_{\alpha} (t) - \alpha \cos \left( 2\varphi_q + \varphi_{e \alpha} - \frac{1}{1 + 2\alpha} \varphi_{\alpha} (t) \right) \right]. \tag{5}
\]

By considering the inherent potential energy of the system and the interaction of the system with the time dependent flux, the potential energy can be written as \[6\]

\[
U = E_j \left( 2 + \alpha - 2 \cos \varphi_p \cos \varphi_q - \alpha \cos (2\varphi_q + \varphi_{e \alpha}) \right) + E_j \frac{\alpha}{1 + 2\alpha} \left( 2 \cos \varphi_p \sin \varphi_q - \sin (2\varphi_q + \varphi_{e \alpha}) \right) \varphi_{\alpha} (t) \tag{6}
\]

Using Eq. (6), the Hamiltonian of the three-junction magnetic flux qubit can be divided into two parts: \( H_0 \) and \( H_i \). The Hamiltonian of the three-junction flux quantum bit is \[7\]

\[
H = H_0 + H_i, \tag{7}
\]

\[
H_0 = \frac{2}{\mu \hbar^2} \left[ P_p + \frac{\Phi_0}{2\pi} Q_{q \varphi_p} \right]^2 + \frac{2}{\mu \hbar^2} \left[ \frac{1 + \gamma}{1 + 2\alpha + \gamma} P_{m} + \frac{\Phi_0}{2\pi} Q_{\alpha m} \right]^2
+ 2 + \alpha - 2 \cos \varphi_p \cos \varphi_q - \alpha \cos (2\varphi_q + \varphi_{e \alpha}) \tag{8}
\]

\[
H_i = \hat{I} \Phi_{e \alpha} (t), \tag{9}
\]

\[
\hat{I} = \frac{2\pi \alpha E_j}{(1 + 2\alpha) \Phi_0} \left( 2 \cos \varphi_p \sin \varphi_q - \sin (2\varphi_q + \varphi_{e \alpha}) \right) \tag{10}
\]

In Eqs. (8)-(10) \( M_m = \frac{1 + 2\alpha + \gamma}{1 + \gamma} \), \( M_p = \frac{1 + 2\alpha + \gamma}{1 + \gamma} \phi \), \( P_p = -i \hbar \frac{\partial}{\partial \varphi_p} \) and \( P_m = -i \hbar \frac{\partial}{\partial \varphi_{\alpha}} \).

When \( \Phi_{e \alpha} (t) = 0 \) \[9\], the quantum properties of the flux qubit are described by \( H_0 \). Normally, the flux \( \varphi_{e \alpha} \) may take a value around 0.5. By ignoring higher energy levels at very low temperatures, a two-level system can be obtained. So the energy level structure of the system can be adjusted by setting the value of \( \varphi_{e \alpha} \).

Thus, the \( H_0 \) of our model is taken to be

\[
H_0 = \frac{E_1 + E_2}{2} I + \frac{E_1 - E_2}{2} \sigma_z, \tag{11}
\]

and satisfy \( \langle \Psi_1 | H_0 | \Psi_1 \rangle = E_1 \), \( \langle \Psi_2 | H_0 | \Psi_2 \rangle = E_2 \), and \( \langle \Psi_2 | H_0 | \Psi_1 \rangle = \langle \Psi_1 | H_0 | \Psi_2 \rangle = 0 \).

When \( \Phi_{e \alpha} (t) \neq 0 \), the flux qubit will interact with the time-dependent magnetic field, and this interaction is described by the Hamiltonian \( H_i \). and \( H_i \) is expressed as

\[
H_i = \sum_g \langle \Psi_i | \hat{I} | \Psi_g \rangle \langle \Psi_g | \Phi_{e \alpha} (t) \rangle \tag{12}
\]

For the lowest three energy levels of the flux qubit, when \( \varphi_{e \alpha} = 0.5 \), the expected value of the current in the three-junction flux quantum bit loop is zero, which can be regarded as a man-made "natural atom" \[10\]. The interacting part of Hamilton can be written as

\[
H_i = \hat{I} \Phi_{e \alpha} (0) (a + a^\dagger). \tag{13}
\]

For the lowest two energy levels,
\[ H_I = \left( \frac{I_1 + I_2}{2} I + \frac{I_2 - I_1}{2} \sigma_z + I_{12}\sigma_x \right) \Phi^{(0)}_{ex}(a + a^+) \]  

(14)

Ignoring the impact of \( \frac{I_2 - I_1}{2} \), let

\[ g_1 = \frac{\Phi^{(0)}_{ex} I_2 - I_1}{2\hbar} \text{ and } g_2 = \frac{\Phi^{(0)}_{ex} I_{12}}{\hbar}. \]

The \( H_I \) is taken to be

\[ H_I = \hbar (g_1 \sigma_z + g_2 \sigma_x)(a + a^+). \]  

(15)

This is the typical interacting term of Rabi model, therefore the Hamiltonian of a three-junction superconducting flux quantum bit system can be quantized into a formation of two-qubit Rabi model.

**Conclusion**

This paper is devoted to studying the quantization of three-junction flux qubit system. When the external magnetic flux consists of a constant part and a time-dependent part, it is found that the Hamiltonian includes the potential energy and the interacting energy caused by the time-varying flux.

Finally, The Hamiltonian of a three-junction superconducting flux quantum bit system be quantized into a formation of two-qubit Rabi model, when the time-dependent magnetic field is different. Based on research works, it has certain significance for understanding the structure of Hamiltonian and the choice of quantum state in experiments.

**References**


