Fast Hankel Transform and Its Application to Beam

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\textbf{Abstract.} In this paper, the numerical technique that is Fast Hankel Transform is presented for the beam propagation in two transverse dimensions with cylindrical symmetry. The Fast Hankel Transform is based on Fastest Fourier Transform in the West, and its application is discussed.

\textbf{Introduction}

In cylindrical coordinates, Hankel transforms are frequency encountered when people study various kinds of waves, electromagnetic field, etc. An efficient algorithm is needed for solving problems associated with this kind of transform.

There are several numerical approaches for implementing the Hankel transform [1-3]. The importance of Siegman’s method [1,2] resides in the fact that, depending on the parameters, one can employ a nonuniform sampling that is denser near the focusing region, which has advantages over uniform sampling. Yu et al. method [3,4], is based on the expansion of the function and its transform by a zero-order Bessel series. The Fast Hankel Transform (FHT) that is based on Fastest Fourier Transform in the West (FFTW) and its application are investigated.

\textbf{Theoretical Model}

The Hankel transform (of order zero) is an integral transform equivalent to a two-dimensional Fourier transform with a radially symmetric integral kernel and also called the Fourier-Bessel transform[5]. It is defined as

\begin{equation}
g(u,v) = F_{\rho} \left[ f(r) \right] (u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r) e^{-2\pi i (ux + vy)} \, dx \, dy
\end{equation}

Let $x + iy = re^{i\theta}$ and $u + iv = qe^{i\phi}$, so that

$x = r \cos \theta$, $y = r \sin \theta$, $r = \sqrt{x^2 + y^2}$, $u = q \cos \phi$, $v = q \sin \phi$, $\rho = \sqrt{u^2 + v^2}$. Then

\begin{equation}
g(\rho) = \int_{-\infty}^{\infty} \int_{0}^{2\pi} f(r) e^{-2\pi i q \cos \phi} \cos \theta \, r \, dr \, d\theta = \int_{0}^{\infty} \int_{0}^{2\pi} f(r) e^{-2\pi i q \cos \phi} \cos \theta \, r \, dr \, d\theta
\end{equation}

\begin{equation}
= \int_{0}^{\infty} \int_{0}^{2\pi} f(r) e^{-2\pi i q \cos \phi} \cos \theta \, r \, dr \, d\theta = \int_{0}^{\pi} \int_{0}^{2\pi} f(r) e^{-2\pi i q \cos \phi} \cos \theta \, r \, dr \, d\theta
\end{equation}

\begin{equation}
= 2\pi \int_{0}^{\infty} f(r) J_0(2\pi r \rho) \, r \, dr
\end{equation}

Therefore, the Hankel transform pairs are[6]
\[
g(\rho) = 2\pi \int_0^\infty f(r)J_0(2\pi r \rho) \rho dr
\]
\[
f(r) = 2\pi \int_0^\infty g(\rho)J_0(2\pi r \rho) \rho d\rho
\]
(3)

Where \( r \) is the radial coordinate; \( \rho \) is the spatial frequency; \( J_0 \) is the zero-order Bessel function of the first kind; \( f(r) \) and \( g(\rho) \) can be either real or complex functions and are of axial symmetry mathematically, representing the field distributions in a spatial domain and spatial frequency domain, respectively.

In the FHT scheme, a Gardner transform variable change is performed

\[
r = r_0 e^{ax}, \quad \rho = \rho_0 e^{ay} \quad [1,2];
\]

The \( f(r) \) and \( g(\rho) \) of the Eq.(3) are defined as

\[
\hat{f}(x) = rf(r) = r_0 e^{ax} f(r_0 e^{ax})
\]
\[
\hat{g}(y) = \rho g(\rho) = \rho_0 e^{ay} g(\rho_0 e^{ay})
\]
\[
j(x) = 2\pi r_0 J_1(2\pi r_0) = 2\pi r_0 \rho_0 e^{a(x+y)} J_1(2\pi r_0 \rho_0 e^{a(x+y)})
\]

Then the transform integrals of Eq.(3) become:

\[
\hat{g}(y) = \int_{-\infty}^{\infty} \hat{f}(x) j(x+y) dx
\]
\[
\hat{f}(x) = \int_{-\infty}^{\infty} \hat{g}(y) j(x+y) dy
\]
(5)

Assuming that \( r_n = r_0 e^{an}, \quad \rho_m = \rho_0 e^{am}, \quad \text{where} \quad \begin{cases} x = n \\ y = m \end{cases} = 0, 1, 2, N \pi .

Thus, \( \hat{f}, \hat{g}, \hat{j} \) are converted into the discrete sequences:

\[
\hat{f}_n = r_n f(r_n) = r_0 e^{an} f(r_0 e^{an})
\]
\[
\hat{g}_m = \rho_m g(\rho_m) = \rho_0 e^{am} g(\rho_0 e^{am}), n,m = 0,1,2,\ldots,N-1
\]
\[
j(q) = 2\pi r_0 \rho_0 e^{a(q-\frac{1}{2})} J_1(2\pi r_0 \rho_0 e^{a(q-\frac{1}{2})}), q = 0, 1, \ldots, 2N-1.
\]

A discrete approximation to the Hankel integrals of Eq.(3) for \( n,m=0,1,\ldots,N-1 \) can be written as:

\[
\hat{g}_m = \sum_{n=0}^{N-1} \hat{f}_n j_{n+m}
\]
\[
\hat{f}_n = \sum_{m=0}^{N-1} \hat{g}_m j_{n+m}
\]
(7)

While \( \hat{f}_n = 0, \hat{g}_m = 0 \) for \( n,m=N,N+1,\ldots,2N-1 \). Thus

\[
\hat{g}_m = FFT^{-1}[FFT^{-1}(\hat{f}_n) \times FFT^{-1}(j_q)]
\]
(8)

Then replacing the FFT by FFTW, the Eq.11) become
The related parameters are \( N = K_2 \beta b \ln(K_1 \beta b), \alpha e^{\alpha m} = K_1 / K_2, r_0 \rho_0 = (K_2 / K_1^2) \alpha \), where \( K_1, K_2 \geq 2 \). Once \( K_1, K_2 \) are given, \( \alpha, r_0, \rho_0, \beta, b \) can be found.

Sheng[2] propound the method of Lower End Correction (LEC) to improve the accuracy as follows.

The part contributed by \( f_{j}(r) \) within the cutoff range to the Hankel transform result is approximately:

\[
\rho \Delta g_j(r) = 2\pi \rho \int_0^\infty \rho f_j(r) J_1(2\pi \rho r) dr \\
= \sum_{q=0}^{\infty} (-1)^q \frac{2^{q+1}}{q!(q+1)!} \int_0^\infty [f_j(0) + f_j'(0)r + \frac{1}{2}f_j''(0)r^2 + \frac{1}{6}f_j'''(0)r^3] \rho^{2q+1} dr \\
= S_{n0} f_j(0) + S_{n1} f_j'(0) + S_{n2} f_j''(0) + ... + S_{nn} f_j^{(n)}(0) + ...
\]

The part of the inaccuracy due to the lower end cutoff can be reduced by add this LEC to the basic FHT.

**Simulation Analysis**

From the figure 1(a) and (b), we can see the precision will increase as the sample number increases based on FHT.
Figure 1. \( A(r) = e^{-r^{1/2}} \) is transformed twice by FHT and compared with itself, (a) N=256, (b) N=2048.

Figure 2 reveals the beam profile by the FHT. The processing results showed that the algorithm has advantages of fast computation, stable and high precision.

**Conclusion**

In conclusion, this paper presents a significant technique to numerically simulate beam propagation based on the NLS equation in two transverse dimensions with cylindrical symmetry. Siegman’s method which was called a FHT in which a fast Fourier transform is used ingeniously, which has good applied prospect in beam propagation. This method is helpful for simulating light bullets which is very important for telecommunication system due to their self confined structure.

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