Investment Opportunity and Investment Value under Symmetrical Duopoly Option Games

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Abstract. This paper considers that two companies in the dual oligopoly market already exist in the market, and the two companies have the same project investment opportunities in the case of how to make investment decisions. Based on the model of Huisman and Kort, we considered that followers obtained discounted cash flow over a certain period of time. Leaders obtained monopoly profits, applied traditional methods of contingent claims analysis, and studied the investment value function and investment criticality of the follower and the leader.

Introduction

In recent decades real options theory has developed many models that can be used to study irreversible investment decisions by firms. These models are mainly based on an analogy between investment projects and the exercise decision of American options.

It is clear, however, that many real options, in contrast to their financial counterparts, are non-exclusive. This implies that there is a game-theoretic component to the analysis of many real options. In the literature, such analyses are usually restricted to games with two firms that exhibit either a first-mover or a second-mover advantage (Grenadier (2000) and Chevalier-Roignant and Trigeorgis (2011) for an overview of the earlier and more recent literature, respectively). In the former case there is preemptive pressure when firms try to be the first to fist, whereas in the latter case there is attrition. Most of the game theoretic real options literature has focused on models with preemption.

At a game theoretic level these papers use the concepts developed by Fudenberg and Tirole (1985) for deterministic timing games. A weakness of much of the current literature is that the results from Fudenberg and Tirole (1985) are applied without much consideration of their appropriateness in a stochastic setting.

The basic problem that is addressed by Fudenberg and Tirole (1985) is that in continuous time it is difficult to model instantaneous reactions before the underlying state variable evolves. This problem is overcome in Fudenberg and Tirole (1985) by using strategies that allow for an “interval of atoms”.

The option game method is an organic combination of real options and game theory. It not only considers the uncertainty, irreversibility, and flexibility of the investment process, but also considers the impact of strategic interaction on investment value in the investment process. It is widely used to analyze the issue of competitive investment in an uncertain environment. Smets (1995) provides a treatment of the duopoly in a multinational setup, which serves as a basis for the oligopoly discussion in Dixit and Pindyck (1994). Grenadier (2002) provides a general solution approach for deriving the equilibrium investment strategies of symmetric firms facing a sequence of investment opportunities with incremental capacity investments. Weeds (2002), Huisman (2001), Huisman and Kort (2003) study option games in a technology adoption context; Boyer et al. (2004) study a duopoly with multiple investments under Bertrand competition; Smit and Trigeorgis (2004) discuss different strategic competition models in the context of real options, while Chevalier-Roignant and Trigeorgis (2011) discuss option games.
The structure of the paper is as follows. In Section 2 we give the model's assumptions. Section 3 discusses the investment value and critical investment of follower companies and the investment value and investment criticality of leading companies. Section 4 concludes.

Model Hypothesis

A symmetric duopoly enterprise game means that there are two competing equivalent companies. Since the enterprises are symmetrical, the leaders and followers in the competition are not given in advance; the two companies have the same rationality; and the two companies do not have asymmetrical balanced cooperation, but only a balanced strategy.

In the dynamic oligopolistic market structure, the two companies on the market are often relatively strong, one relatively weak, and rare situations where competitors are difficult to distinguish. The weaker companies often make decisions later, that is, they first observe the actions of the strong. Here, the party who performed the action is referred to as the leader, and the party who performs the action is called the follower. In a certain period of time, the demand of the entire market remains stable and the overall demand is constant. The market share occupied by leaders will change with the followers' access, and the profits of the leaders will also be affected. Therefore, when a leader company discovers that potential followers have entered the market, it is necessary to fully consider the investment strategies that followers may adopt when making decisions on their own companies.

**Hypothesis 1** The assumption of complete symmetry. Assume that in the duopoly market, there are two completely symmetric companies, which are risk-neutral, all seek to maximize the expected value, and compete with each other.

**Hypothesis 2** Assumption of uncertainty. Assuming that the two companies only have market uncertainty, the investment project itself can be successful once it starts to invest, and the technology is mature and reliable. The project investment started by one party will affect the investment income and investment decisions of the other party.

**Hypothesis 3** Assume that two companies already exist in the market, no new ones enter, and the two companies are already operating the investment project. Since the company is risk-neutral, the model is discounted at a fixed risk-free rate, and the discounted moment is the optimal investment opportunity for the leader.

**Hypothesis 4** assumes that the investment cost for the new project is the same as the initial investment fund, i.e., the strike price of the option.

**Hypothesis 5** assumes that the inverse demand function of the firm is $P(t)$

$$P(t) = Y(t)D(N_i, N_j)$$

(1)

$Y(t)$ represents the random market demand impact factor, assuming that it obeys the following geometric Brownian motion:

$$dY_t = \mu Y_t dt + \sigma Y_t d\omega(t) \quad Y > 0, \ Y > 0, \ 0 < \mu < \sigma, \ \sigma > 0, \ d\omega \sim N(0, dt)$$

(2)

Where $\mu$ is the drift term, $\sigma$ is the volatility, and $d\omega(t)$ is the normal distribution of $dt$. In the following discussion, a unified description of the variable's representation and notation is used. Instead of causing confusion, $Y$ is used instead of $Y(t)$. Assuming the moment of occurrence, the market demand is 1, that is, $Y(0)=1$.

$D(N_i, N_j)$ represents the determined market demand parameter, which depends on the investment of the company $i$ and $j$.

$$N_k = \begin{cases} 0, & \text{if company } k \text{ does not invest} \\ 1, & \text{if company } k \text{ has already invested} \end{cases} \quad k \in \{i, j\}$$

Therefore, $D(N_i, N_j)$ can be used to indicate the mutual status and competitive relationship between two companies.
Assume that the investment in the new project has negative externalities. The value of an enterprise's project investment (option value) or profit flow will decrease because of the entry of competitors. So for companies, the following restrictions must be imposed, given by the following inequalities:

\[ D(1,0) > D(1,1) > D(0,0) > D(0,1) > 0 \]  \quad (3)

In addition, the market parameters also make the following assumptions: The company has a first-mover advantage when investing and meets the following conditions:

\[ D(1,0) > D(0,0) > D(1,1) > D(0,1) \]  \quad (4)

First-mover advantage is a comparative benefit, which means that when investing in a project, the preemptive party becomes the leader and can gain the preemptive advantage. The preemptor gains more income than the follower after the project investment.

**Hypothesis 6** Assuming that the project investment opportunity is the same for both companies, the investment opportunity is shared by the two companies. The time from the start of investment to the completion of the project is \( \delta \) years, i.e. the investment project construction period. For example, if an enterprise invests at \( \tau \), then it can only make a profit flow at \( (\tau + \delta) \).

**The Model**

**The Follower's Investment Value and Investment Criticality**

The two companies in the dual oligopoly market considered in this article already exist in the market and belong to the original market model. The two companies have the same investment opportunities for the project. In order to follow the investing value function and investment criticality of followers, we first assume that one of them is that the leader has started the investment, and then the follower's investment value function is sought. The follower's investment value \( F(Y) \) can be expressed as the combination of the discounted cash flow \( YD(0,1)e^{-(r-\mu)\delta} \) obtained by the follower through the delta period and the value of the execution investment option. Technically, we apply the traditional method of contingent claims analysis (Dixit and Pindyck, 1994). Using the Itô lemma, the follower's investment value \( F(Y) \) satisfies the following second-order differential equation:

\[
\frac{1}{2} \sigma^2 Y^2 F_{yy} + \mu Y F_y - rF + YD(0,1)e^{-(r-\mu)\delta} = 0
\]  \quad (5)

Compared with the model of Huisman and Kort (1999), the constant term in equation (5) is different and there is one more factor \( e^{-(r-\mu)\delta} \). The reason is that we assume that two companies already exist in the market, and that the profit flow before the two companies have not started project investment is \( YD(0,0) \), assuming that the leader has started to invest, and the completion time of the investment project is \( \delta \) because of being led. With the impact of preemptive investment, the profit flow of followers will change and the discounted profit flow will change from \( YD(0,0) \) to \( YD(0,1)e^{-(r-\mu)\delta} \).

When seeking the investment value of the follower's company, for the convenience of the solution and the simplicity of the model, the following assumptions are added. Assuming that the leader has already started investing at the initial time \( t=0 \), the profit flow of the follower’s company will be immediately affected; if the leader’s company invests in \( T_L \), then the profit flow of the follower’s company is \( YD(0,0) \) in the interval \((0, T_L)\). In the interval \((T_L, T_F)\) is \( YD(0,1)e^{-(r-\mu)\delta} \). Followers start investing in new projects and participate in market competition at \( T_F \). The profit flow turns into \( YD(1,1)e^{-(r-\mu)\delta} \). \( T_L \) and \( T_F \) respectively represent the optimal investment time for the leader and the follower.
From the structure of solutions of second-order ordinary differential equations, we can see that the solution of (5) is composed of a general solution $AY^β + BY^β$ of the corresponding homogeneous equation plus a special solution of the equation, where $β_1$, $β_2$ are the two roots of the characteristic equation (6).

$$\frac{1}{2}\sigma^2β^2 + (\mu - \frac{1}{2}\sigma^2)β - r = 0$$

(6)

Solving equations (1.6), thus

$$β_1 = \frac{-(\mu - 0.5\sigma^2) + \sqrt{(\mu - 0.5\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} > 1$$

$$β_2 = \frac{-(\mu - 0.5\sigma^2) - \sqrt{(\mu - 0.5\sigma^2)^2 + 2r\sigma^2}}{\sigma^2} < 0$$

Since 0 is the absorption wall of geometric Brownian motion, when the market demand $Y$ of the product is zero, the follower's investment value is equal to zero, and then the negative root $β_2$ is discarded and the positive root $β_1$ is left (see Dixit and Pindyck, 1994).

The solution to the second-order differential equation (5) is:

$$F(Y) = \begin{cases} 
AY^β + \frac{YD(0,1)e^{-(r-\mu)δ}}{r-\mu}, & Y < Y_F \\
YD(1,1)e^{-(r-\mu)δ} - I, & Y \geq Y_F 
\end{cases}$$

(7)

Here, $Y_F$ is the optimal investment criticality of the follower's company and is determined by the following boundary conditions:

$$F(Y_F) = Y_F D(1,1)e^{-(r-\mu)δ} - I = AY_F^β + \frac{Y_F D(0,1)e^{-(r-\mu)δ}}{r-\mu}$$

(8)

$$F_Y(Y_F) = \frac{D(1,1)e^{-(r-\mu)δ}}{r-\mu} = AβY_F^{-1} + \frac{D(0,1)e^{-(r-\mu)δ}}{r-\mu}$$

(9)

Among them, (8) is a value matching condition, and its meaning is that the follower's investment value function is equal to the termination return when the known follower enterprise reaches the optimal investment threshold; (9) is a smooth paste condition indicates that the follower's investment value function is equal to the known function value of the follower's firm when it reaches the optimal investment threshold, and the derivatives or slopes of the two functions are also equal at the boundary. $Y_F$ and $A$ can be calculated from simultaneous equations (8), (9), and (3):

$$Y_F = \frac{β - (r-\mu)e^{-(r-\mu)δ}I}{β - 1 D(1,1) - D(0,1)}$$

$$A = \frac{Y_F^{1-β} [D(1,1) - D(0,1)] e^{-(r-\mu)δ}}{β(r-\mu)} > 0$$

Substituting $A$ and $Y_F$ into (7) gives the follow-up company's investment value function:
Through the above analysis, the optimal investment opportunity for followers can be obtained. The following conclusions can be drawn.

**Proposition 1** Assume that in the duopoly market, after the leader has made the optimal decision for an investment project shared by the duopoly, the best investment strategy for the follower company for the project is to invest at time \( T_f = \min\{t \geq 0 : Y(t) \geq Y_f \} \).

In the same analysis, the following conclusions can be drawn by investing in two companies simultaneously.

**Proposition 2** In the duopoly market, if two investment companies simultaneously invest in an investment project shared by a duopoly company, then the value of any company's investment is:

\[
S(Y) = \frac{YD(1,1)e^{-(r-\mu)\delta}}{r-\mu} - I
\]  

(11)

The Leader's Investment Value and Investment Criticality

From the above discussion, assuming that the leader has executed the investment option, followers have adopted the optimal investment strategy. The leader has already started to invest, and the investment project will take a long time to complete. During this time, the leader will not bring any additional profits during the construction of the investment project. Before the market demand reaches the critical point of follower's investment \( Y < Y_f \), followers will not make investments. At this time, the monopoly return of the leader is \( YD(1,0)e^{-(r-\mu)\delta} \). However, when the follow-up company performs project investment after executing real options, the market income of the leader is reduced due to the participation of competitors, and the leader's income is reduced from the monopoly \( YD(1,0)e^{-(r-\mu)\delta} \) to \( YD(1,1)e^{-(r-\mu)\delta} \). For leading companies, the leading investment project (shared by the duopoly companies) can be seen as a real option (option for project investment), and the value of the underlying asset (the investment project) is \( V(Y) \). The price is \( I \). The leading company value is \( L(Y) = V(Y) - I \). The leader's strategic decision-making is to find the best investment opportunity to invest in new projects. Similarly, by the Itô lemmas, the entire project value \( V(Y) \) satisfies the following second-order differential equations:

\[
\frac{1}{2} \sigma^2 Y^2 V_{YY} + \mu Y V_Y - rV + YD(1,0)e^{-(r-\mu)\delta} = 0
\]  

(12)

Compared with formula (5), the difference lies in the final item of the differential equation. Since the followers have not yet begun to invest at this time, the leader has exclusive monopoly on the profit flow of the investment project \( YD(1,0)e^{-(r-\mu)\delta} \). Similar to the above solution, the solution of (12) can be written as

\[
V(Y) = BY^{\beta} + YD(1,0)e^{-(r-\mu)\delta} \frac{r-\mu}{r-\mu}
\]  

(13)

Contrast formulas (7) and (13), where the undetermined constant \( B \) in (13) is negative, the economic meaning of the expression is that when the follower enters the market and begins to invest, the market income of the leader decreases, and the profit from the monopoly stream \( YD(1,0)e^{-(r-\mu)\delta} \)
reduces profit flow to competition \( YD(1,1)e^{-(r-\mu)\delta} \). The follower's business is at the investment critical value \( Y_F \), and its project value function is \( \frac{Y_F D(1,1)e^{-(r-\mu)\delta}}{r-\mu} \).

From this boundary condition, then, \( V(Y_F) = \frac{Y_F D(1,1)e^{-(r-\mu)\delta}}{r-\mu} \).

According to the value matching conditions, it can be solved

\[
B = \frac{Y_F D(1,1) - D(1,0)}{Y_F^\beta} < 0
\]

\( B < 0 \) just explains that when the follower companies enter the market and start investing, the market return of the leader is reduced. By putting the expression into (13), the investment value of the leader company is obtained,

\[
L(Y) = \begin{cases}
\frac{Y}{Y_F (\beta-1)[D(1,1) - D(0,1)]} & Y < Y_F \\
\frac{YD(1,1)e^{-(r-\mu)\delta}}{r-\mu} - I, & Y \geq Y_F
\end{cases}
\]

(14)

Here, the solution to the leader's investment value function and investment criticality is not the same as the follower's investment value function and investment critical situation, and smooth boundary conditions with equal boundaries have not been used. The reason is that the optimal investment margin for follower companies is not optimal for leading companies. Assuming that the two companies in the oligopolistic market are completely symmetrical, the status of the preemptive and followers is also equal to them, but because preemptive investments can yield temporary monopolistic returns, they have to obtain temporary monopoly gains. Motivation, then there will be two companies have to compete for investment, the result is that everyone's investment time ahead of schedule. Eventually, it will reach a state of equilibrium. Under balanced conditions, both parties will not behave in a way that preempts their opponent's investment. At this time, the investment value of the leader and the follower is equal. Therefore, the leading edge of investment is

\[
Y_L = \min\{Y \mid L(Y) = F(Y), 0 < Y < Y_F\}
\]

(15)

Then, we need to calculate the follower's investment threshold. According to the above analysis, because the two companies will have advanced investment in order to obtain excess returns, the final equilibrium state is that both parties will not behave in a preemptive manner. The investment value of the leader is equal to the investment value of the follower. The follower's investment criticality can be determined by the relative investment value of the leader and the follower. Let the relative value function be \( \xi(Y) \), then \( \xi(Y) = L(Y) - F(Y) \). In the interval \((0, Y_F)\), we discuss the value of \( \xi(Y) \):

If \( Y = 0 \), \( \xi(Y) = -I < 0 \);

If \( Y = Y_F \), \( \xi(Y) = 0 \).

As \( Y \to Y_F \), \( \left( \frac{Y}{Y_F} \right)^{\beta-1} \to 1 \), then \( \lim_{Y \to Y_F} \xi(Y) = \frac{e^{-(r-\mu)\delta}}{r-\mu} (\beta-1)[D(1,1) - D(1,0)] < 0 \), we get \( \xi(Y) \) has at least one root in interval \((0, Y_F)\).

And \( \xi^\prime(Y) = -\frac{\beta Y}{Y^2} \left( 1 + \beta \frac{D(1,1) - D(1,0)}{D(1,1)} \right) \left( \frac{Y}{Y_F} \right)^\beta I < 0 \)

From the above analysis, the following conclusions can be drawn.
Proposition 3 There has uniqueness $Y_L$ in the interval $(0, Y_F)$, when $Y < Y_L$, $L(Y) < L(Y_L)$; when $Y < Y_L$, $L(Y) = L(Y_L)$; when $Y < Y_L$, $L(Y) > L(Y_L)$; when $Y \geq Y_L$, $L(Y) = L(Y_L)$.

Proposition 4 Assume that the follower invests at the optimal investment time $T_F = \min \{ t \geq 0 : Y(t) \geq Y_F \}$. The leader already knows the follower's optimal investment strategy and optimal investment timing. The leader's investment threshold is $Y_L$, then, the investment time of the leader's optimal investment strategy is $T_L = \min \{ t \geq 0 : L(Y_L) = F(Y_L) \}$.

Conclusion
In this article we discuss two companies in the dual oligopoly market already exist in the market, both companies have the same project investment opportunities. Based on the model of Huisman and Kort, they consider followers and leaders to obtain certain cash flow or monopoly profits. Afterwards, the traditional method of analysis of contingent claims was applied to provide decisions for followers and leaders.

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