Stability of Exponential Euler Method for Linear Stochastic Delay Differential Equations

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Abstract. The main purpose of this paper is to investigate the exponential stability in mean square of the exponential Euler method to linear stochastic delay differential equations (LSDDEs). The classical stability theorem to LSDDEs is given by the Lyapunov functions. However, in this paper we study the exponential stability in mean square of the exact solution to LSDDEs by using the definition of logarithmic norm. On the other hand, the implicit Euler scheme to LSDDEs is known to be exponentially stable in mean square for any step size. However, in this article we propose an explicit method to show that the exponential Euler method to LSDDEs is proved to share the same stability for any step size by the property of logarithmic norm.

Introduction

Stochastic modeling has come to play an important role in many branches of science and industry. Such models have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, economics and finance. Most stochastic differential equations (SDEs) are nonlinear and cannot be solved explicitly, whence numerical solutions are required in practice. Numerical solutions to SDEs have been discussed under the Lipschitz condition and the linear growth condition by many authors (see [1],[2],[3],[4]). Many authors have discussed numerical solutions to stochastic delay differential equations (SDDEs) (see [5],[6],[7]). The stability of the implicit Euler scheme to SDEs is known for any step size. However, in this article we propose an explicit method to show that the exponential Euler method to LSDDEs is proved to share the same stability for any step size by the property of logarithmic norm.

Preliminary Notation and the Exponential Euler Method

Let $B(t) = (B_1(t),\cdots,B_d(t))^T$ be a $d$-dimensional Brownian motion defined on the probability space $(\Omega,F,P)$. Throughout this paper, we consider the following linear stochastic delay differential equations:

$$\begin{cases}
  dx(t) = (Ax(t) + Bx(t - \tau))dt + (Cx(t) + Dx(t - \tau))dB(t), t \in [0,T] \\
  x(t) = \xi(t), t \in [-\tau,0],
\end{cases}$$

(1)

where $T > 0, \tau > 0, \{\xi(t), t \in [-\tau,0]; \mathbb{R}^n\}, A \in \mathbb{R}^{n \times n}$ the matrix [8], $B, C, D$ are constants. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation:

$$x(t) = e^{At} \xi + \int_0^t e^{A(t-s)} (Ax(s) + Bx(s - \tau))ds + \int_0^t e^{A(t-s)} (Cx(s) + Dx(s - \tau))dB(s), \forall t \geq 0.$$  

(2)
Let $h = \frac{\tau}{m}$ be a given step size with integer $m \geq 1$; and let the grid points $t_n$ be defined by $t_n = nh(n = 0, 1, 2, \cdots)$. We consider the exponential Euler method to \( (1) \)

\[ y_{n+1} = e^{Ah} \xi + e^{Ah}(Ay_n + By_{n-m})h + e^{Ah}(Cy_n + Dy_{n-m})\Delta B_n, \]  

where $\Delta B_n = B(t_n) - B(t_{n-1})$, $n = 0, 1, 2, \cdots$, $y_n$ is approximation to the exact solution $x(t_n)$. The continuous exponential Euler method approximate solution is defined by

\[ y(t) = e^{At} \xi + \int_0^t e^{A(t-s)}(Az(s) + Bz(t - \tau))ds + \int_0^t e^{A(t-s)}(Cz(t) + Dz(t - \tau))dB(s), \forall t \geq 0. \]  

where $z(t) = \sum_{k=0}^{\infty} y_k I_{\{kh, (k+1)h\}}(t)$ with $I_A$ denoting the indicator function for the set $A$. It is not difficult to see that $y(t_n) = z(t_n) = y_n$ for $n = 0, 1, 2, \cdots$. That is, the step function $z(t)$ and the continuous exponential Euler solution $y(t)$ coincide with the discrete solution at the grid point.

**Exponential Stability in Mean Square**

In this section, we give the exponential stability in mean square of the exact solution and the exponential Euler method to semi-linear stochastic delay differential equations (1).

**Stability of the Exact Solution**

In this subsection, we will show the exponential stability in mean square of the exact solution to semi-linear stochastic delay differential equations (1). Next we will give the main content of this subsection.

**Theorem 3.1.** If $1 + 2\mu[A] + 4\left( B^2 + C^2 + D^2 \right) < 0$, then the solution of equations (1) with the initial data $\xi \in C_{F_0}^b([-\tau, 0], R^n)$ is exponentially stable in mean square, that is,

\[ E \left| x(t) \right|^2 \leq \tilde{B}^{-1}(\tau)E \left| \xi \right|^2 e^{\mu I(t)\tau^2}, t \geq 0, \]

where $\tilde{B}(\tau) = e^{B\tau} - \frac{B_2}{B_1}(1 - e^{B\tau})$, $B_1 = 1 + 2\mu[A] + 2(B^2 + C^2 + D^2)$, $B_2 = 2(B^2 + C^2 + D^2)$.

By Ito’s formula and the delay term of the equation, we give the proof of Theorem 3.1. The highlight of the proof is that we give the mean square boundedness of the solution to the equation by dividing the interval into $[0, \pi], [\pi, 2\pi], \cdots, [k\pi, (k+1)\pi]$. Then we give a proof of the conclusion by $t \geq 0$, $t \geq 2\pi$, $t \geq 4\pi$, $\cdots$, $t \geq 2n\pi$. In the process of dealing with the semi-linear matrix, we use the definition of the matrix norm.

**Definition 3.1.** [7] SDDes (1) are said to be exponentially stable in mean square if there is a pair of positive constants $\lambda$ and $\mu$ such that for any initial data $\xi \in C_{F_0}^b([-\tau, 0], R^n)$

\[ E \left| x(t) \right|^2 \leq \mu E \left| \xi \right|^2 e^{-\lambda t}, t \geq 0. \]

We refer to $\lambda$ as the rate constant and to $\mu$ as the growth constant.

**Definition 3.2.** [9] The logarithmic norm $\mu[A]$ of $A$ is defined by $\mu[A] = \lim_{\Delta \to 0} \| I + \Delta A \|^{-1}. \Delta$

Especially, if $\| \cdot \|$ is an inner product norm, $\mu[A]$ can also be written as
\[ \mu[A] = \max_{\xi \neq 0} \frac{\langle A\xi, \xi \rangle}{\|\xi\|^2}. \]  

(5)

**Lemma 3.1.** Let \[ \hat{B}(t) = e^{B_t} - \frac{B_2}{B_1} (1 - e^{B_t}). \] If \( B_1 < 0, B_2 > 0 \) and \( B_1 + B_2 < 0 \) then for all \( t \geq 0, 0 < \hat{B}(t) \leq 1 \) and \( \hat{B}(t) \) is decreasing.

**Proof.** It is known from \( B_1 < 0, B_2 > 0 \) and \( B_1 + B_2 < 0 \) then for all \( t \geq 0, \) \( B(t) = \frac{B_1 + B_2}{B_1} e^{B_t} - \frac{B_2}{B_1} > 0 \)

and

\[ \hat{B}(t) = e^{B_t} - 1 + \frac{B_2}{B_1} (e^{B_t} - 1) + \frac{(B_1 + B_2)(e^{B_t} - 1)}{B_1} + 1 \leq 1. \]

For all \( t \geq 0, \) we compute \( \hat{B}(t) = (B_1 + B_2) e^{B_t} < 0. \) Thus \( \hat{B}(t) \) is decreasing. The proof is complete.

**Proof of Theorem 3.1.** By Ito’s formula and Definition 3.2, for all \( t \geq 0; \) we have

\[ d| x(t) |^2 = \left[ < 2x(t), Ax(t) + Bx(t - \tau) > + | Cx(t) + Dx(t - \tau) |^2 \right] dt \
\quad + 2x^T(t)(Cx(t) + Dx(t - \tau)) dB(t) \
\quad \leq [B_1 | x(t) |^2 + B_2 | x(t - \tau) |^2] dt + 2x^T(t)(Cx(t) + Dx(t - \tau)) dB(t). \]  

(6)

Where \( B_1 = 1 + 2\mu[A] + 2(B^2 + C^2 + D^2), B_2 = 2(B^2 + C^2 + D^2) \). Let \( V(x, t) = e^{-B_t} | x(t) |^2 \), by Ito’s formula, we obtain

\[ d| e^{-B_t}x(t) |^2 = -B_1 e^{-B_t} | x(t) |^2 dt + e^{-B_t} d| x(t) |^2 \leq B_2 e^{-B_t} | x(t - \tau) |^2 dt \
\quad + 2e^{-B_t} x^T(t)(Cx(t) + Dx(t - \tau)) dB(t) \]

(7)

Integrating (7) from 0 to \( t \) and taking expected values gives

\[ e^{-B_t} E | x(t) |^2 = E | x(0) |^2 + B_2 \int_0^t e^{-B_s} E | x(s - \tau) |^2 ds. \]

For any \( t \in [0, \tau], \) we have

\[ E | x(t) |^2 \leq [e^{B_t} - \frac{B_2}{B_1} (1 - e^{B_t})] E | \xi |^2 = B(t) E | \xi |^2. \]

For any \( t \in [\tau, 2\tau], \) we have

\[ E | x(t) |^2 \leq [e^{B(t-\tau)} - \frac{B_2}{B_1} (1 - e^{B(t-\tau)})] E | \xi |^2 = B(t-\tau) E | \xi |^2. \]

Repeating this procedure, for all \( t \in [k\tau, (k+1)\tau], \) we can show

\[ E | x(t) |^2 \leq B(t-k\tau) E | \xi |^2. \]

So, for any \( t \geq 0; \) we have \( E | x(t) |^2 \leq E | \xi |^2. \) On the other hand, for any \( t \geq 0, \) we obtain
\[ E \left| x(t) \right|^2 \leq \left[ e^{B(t)} - \frac{B_2}{B_1} (1 - e^{B(t)}) \right] E \left| \xi \right|^2 = \bar{B}(t) E \left| \xi \right|^2. \]

Especially, we can see
\[ E \left| x(2 \tau) \right|^2 \leq \bar{B}(2 \tau) E \left| \xi \right|^2. \]

For any \( t \geq 2 \tau \), we have
\[ E \left| x(2 \tau) \right|^2 \leq \bar{B}(\tau) \bar{B}(2 \tau) E \left| \xi \right|^2 \leq \bar{B}^2(\tau) E \left| \xi \right|^2. \]

For any \( t \geq 4 \tau \), we can see that
\[ E \left| x(t) \right|^2 \leq \bar{B}^2(\tau)[ e^{B(t-4 \tau)} - \frac{B_2}{B_1} (1 - e^{B(t-4 \tau)}) ] E \left| \xi \right|^2 = \bar{B}^2(\tau) \bar{B}(t-4 \tau) E \left| \xi \right|^2. \]

For any \( t \geq 0 \) there is an integer \( n \) such that \( t \geq 2n \tau \), repeating this procedure, we can show
\[ E \left| x(t) \right|^2 \leq \bar{B}^n(\tau) \bar{B}(t-n \tau) E \left| \xi \right|^2 \leq \bar{B}^n(\tau) E \left| \xi \right|^2. \]

By (8) and Lemma 3.1, we obtain
\[ E \left| x(t) \right|^2 \leq \bar{B}^n(\tau) E \left| \xi \right|^2 \]
\[ = e^{2n \tau \ln(\bar{B}(\tau)^{\frac{1}{n}})} E \left| \xi \right|^2 \]
\[ = e^{(2n \tau - t) \ln(\bar{B}(\tau)^{\frac{1}{n}})} E \left| \xi \right|^2 e^{n \tau \ln(\bar{B}(\tau)^{\frac{1}{n}})} \]
\[ \leq e^{-2 \tau \ln(\bar{B}(\tau)^{\frac{1}{n}})} E \left| \xi \right|^2 e^{n \tau \ln(\bar{B}(\tau)^{\frac{1}{n}})} \]
\[ \leq \bar{B}^{-1}(\tau) E \left| \xi \right|^2 e^{n \tau \ln(\bar{B}(\tau)^{\frac{1}{n}})}, \]

which proves the theorem.

**Stability of the Exponential Euler Method**

In this subsection, under the same conditions as those in Theorem 3.1, we will obtain the exponential stability in mean square of the exponential Euler method (4) to LSDDEs (1) in Theorem 3.2.

**Definition 3.3.**[7] Given a step size \( h = \frac{\tau}{m} \) for some positive integer \( m \), the discrete exponential Euler method is said to be exponentially stable in mean square on SDDEs (1) if there is a pair of positive constants \( \bar{\lambda} \) and \( \bar{\mu} \) such that for any initial data \( \xi \in C_{F_0}([-\tau, 0], R^n) \),
\[ E \left| y_n \right|^2 \leq \bar{\mu} E \left| \xi \right|^2 e^{-\tau n h}, \quad n \geq 0. \]

**Theorem 3.2.** If \( 1 + 2\mu[A] + 4 \left( \bar{a}^2 + \bar{c}^2 + \bar{d}^2 \right) < 0 \) then for all \( h > 0 \) the numerical method to equations (1) is exponentially stable in mean square, that is
\[ E \left| y_n \right|^2 \leq (A_1 + A_2)^{-1} E \left| y_0 \right|^2 e^{n \tau \ln(\bar{A}_1 + \bar{A}_2)^{\frac{1}{n}}}, \]

Where
\[ A_1 = e^{2\tau(A_1 h) (1 + \bar{a}^2 + \bar{c}^2 + \bar{d}^2) h^2 + 2(\bar{b}^2 + \bar{c}^2 + \bar{d}^2) h)}, \]
\[ A_2 = e^{2\tau(A_1 h) (\bar{b}^2 + \bar{c}^2 + \bar{d}^2) h^2 + 2(\bar{b}^2 + \bar{c}^2 + \bar{d}^2) h)}. \]

**Proof.** Squaring and taking the conditional expectation on both sides of (3), noting that \( \Delta B_n \) is independent of \( F_{nh} \), \( E(\Delta B_n | F_{nh}) = E(\Delta B_n) = 0 \) and \( E((\Delta B_n)^2 | F_{nh}) = E(\Delta B_n)^2 = h \), we have
Taking expectations on both sides, we obtain that
\[ E \left( | y_n |^2 | F_{n+h} \right) = e^{2 \mu(A)h} E \left( | y_n |^2 + e^{2 \mu(A)h} E \left( Ay_n + By_{n-m} |^2 | F_{nh} \right) h^2 + e^{2 \mu(A)h} E \left( Cy_n + Dy_{n-m} |^2 | F_{nh} \right) h^2 \right) + 2 e^{2 \mu(A)h} E \left( < y_n, Ay_n + By_{n-m} > | F_{nh} \right) h^2, \]

Where
\[ A_1 = e^{2 \mu(A)h} (1 - 4(B^2 + C^2 + D^2)h^2 + 2(B^2 + C^2 + D^2)h + h), \]
\[ A_2 = e^{2 \mu(A)h} (B^2 + C^2 + D^2)h^2 + 2(B^2 + C^2 + D^2)h). \]

for all \( h > 0 \); which implies
\[ 1 + h + 4(B^2 + C^2 + D^2)h + 2(B^2 + C^2 + D^2)h^2 < 1 - 2 \mu(A)h + \frac{(-2 \mu(A)h)^2}{2!} < e^{2 \mu(A)h}. \]

That is \( A_1 + A_2 < 1 \) for all \( h > 0 \). For all \( n = 1, 2, \ldots \), we have
\[ E \left( | y_n |^2 \right) \leq A_1 E \left( | y_n |^2 \right) + A_2 E \left( | y_{n-m} |^2 \right) \leq (A_1 + A_2)^{\frac{1}{m+1}} E \left( | y_0 |^2 \right) \leq e^{(\frac{1}{m+1}) \ln(A_1 + A_2)} E \left( | y_0 |^2 \right) e^{\frac{\ln(A_1 + A_2)}{(m+1)}} \]
\[ = (A_1 + A_2)^{-\frac{1}{m+1}} E \left( | y_0 |^2 \right) e^{h \ln(A_1 + A_2) \frac{1}{(m+1)}}. \]

The proof is completed.

Numerical Experiments

In this section, we give several numerical experiments in order to demonstrate the results about the strong convergence and the exponential stability in mean square of the numerical solution for equations (1). We consider the test equation
\[ dx(t) = [a_1 x(t) + a_2 x(t - \tau)] dt + [b_1 x(t) + b_2 x(t - \tau)] dB(t) \quad \forall t \geq 0 \]

\[ (9) \]

Example 9. When \( a_1 = -5, a_2 = 1, b_1 = 2, b_2 = 0.5, \xi = 1 + t, \tau = 1. \) We can show the stability of the exponential Euler method to (3). In Figure 1, all the curves decay toward to zero when \( h = \frac{1}{2}, h = \frac{1}{8}, h = \frac{1}{32}, h = \frac{1}{128}. \) So we can consider that our experiments are consistent with our proved results in Section 3.
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References


