Numerical Methods for Fredholm Integral Equations of the First Kind

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Abstract. In the engineering and scientific fields, many problems can be mathematically described by Fredholm integral equations of the first kind, which have a prominent feature called as “ill-posedness”. This feature makes it extremely difficult to obtain the analytical solution. In recent years, various numerical methods have been commonly developed by both international and domestic scholars to solve this kind of equations. In this paper, the mainstream of current numerical methods for Fredholm integral equations of the first kind have been reviewed and evaluated in details, then a guideline of using the intelligence algorithms to solve the Fredholm integral equations of the first kind is provided in the end.

Introduction

Mathematics and computational science have been widely used to solve the complicated problems in various application fields. Fredholm integral equations of the first kind have been arisen in many fields such as engineering, physics, mechanics, geological exploration, etc.

A Fredholm equation is an integral equation in which the term containing the kernel function has constants as integration limits. It is well-known that the most prominent feature of the Fredholm integral equation of the first kind is ill-posedness, which makes it extremely difficult to obtain its analytical solution. The vast majority of Fredholm integral equations of the first kind can only be solved via the use of numerical methods. In recent years, various numerical methods have been commonly developed by both domestic and international scholars to solve Fredholm integral equations of the first kind.

In this paper, the numerical methods developed for solving Fredholm integral equations of the first kind have been reviewed and evaluated in details. The existence and convergence of the solution of the Fredholm integral equation of the first kind is analyzed in Section 2, the determining theorems and the current research on existence and convergence of the solution are presented. In the next section, the mainstream of current numerical methods of the Fredholm integral equation of the first kind such as the regularization method, the wavelet analysis method and the multilevel iteration method, are discussed and evaluated in details. Finally, a concise overview of these numerical methods is presented, and a guideline of using the intelligence algorithms to solve the Fredholm integral equations of the first kind is provided in the end.

Analysis of Existence and Convergence of the Solution of the Fredholm Integral Equation

A Fredholm equation is an integral equation in which the term containing the kernel function has constants as integration limits. A Fredholm equation of the first kind can be written as:

\[ f(x) = \int_a^b K(x, t)g(t)dt \]

and the problem is, given the continuous kernel function \(K(x, t)\) and the function \(f(x)\), to find the function \(g(t)\).
Determining Theorems of Existence and Convergence of Solutions of Fredholm Equations

Existence and convergence of the solution for a Fredholm equation need to be determined via the following theorems.

Theorem 2.1. Let \( \{ \lambda_i \} \) and \( \{ \phi_i(x), \phi_i(x) \} \) be the eigenvalues and the eigenfunctions of the Fredholm integral equation of the first kind, and the orthonormal system \( \{ \phi_i(x) \} \) is complete for a square-integrable function, then the necessary and sufficient condition for the Fredholm integral equation to have a solution is that the series \( \sum_{i=1}^{\infty} \lambda_i^2 \| f_i \|^2 \) is converged, where \( f_i = (f, \phi_i) \).

Theorem 2.2[1]. When the Fredholm integral equation of the first kind has a solution, then the sequence \( K g_n \) is uniform convergent to \( f(x) \) if and only if the orthonormal system \( \{ \phi_i(x) \} \) is complete for \( f(x) \); the sequence \( g_n(x) \) is uniform convergent to \( g(x) \) if and only if the orthonormal system \( \{ \phi_i(x) \} \) is complete for \( g(x) \).

Theorem 2.3[2]. When the solution \( g \) of the Fredholm integral equation of the first kind exists, and \( g_0 = K^* h_0, h_0 \in L_2[a,b] \) then (a) \( g_n \) is mean-square convergent to \( g \) if and only if \( \{ \phi_i(x) \} \) is complete for \( g \); (b) \( g_n \) is uniform convergent to \( g \) if and only if \( g = K^* h, h \in L_2[a,b] \).

Based on these theorems, one can tell whether a Fredholm equation of the first kind has a solution or not, and whether the solution is convergent or divergent.

Research on Existence and Convergence of Solutions of Fredholm Equations

In recent years, more and more research has been developed on the existence and convergence of the solutions of Fredholm equations of the first kind.

C. W. Groetsch[3] presented an asymptotic convergence analysis of a regularized degenerate kernel method for Fredholm integral equations of the first kind. Convergence theorems were proved in both the mean-square and uniform norms and the Tikhonov-regularity of the method for the case of inexact data was established in both norms. P. Rajan[4] considered a modified convergence analysis for solving Fredholm integral equations of the first kind in Hilbert space setting using Tikhonov regularization. He followed a general approach which not only includes, as special case, the results of Groetsch but also obtains the same results with weaker assumptions. Khosrow Maleknejad et al.[5] studied the Fredholm equation of the first kind with a degenerate kernel which has bounded inverse. The collocation method by wavelet families was utilized for numerical solving of the equation and the convergence of this numerical solution was proven. Then the conjugate gradient method was used for solving the system of linear equations after discretizing the integral equation.

Qing-ming Zhang et al.[6] employed the fixed-point theorem to investigate the existence of the positive solution of the Fredholm equation with the form of \( u(t) = \int_0^1 k(t,s) f(s, u(s)) ds \). Results showed that the positive solution of the Fredholm equation does exist as long as the nonlinear term remains a proper increasing speed. Ming-hui Liu[7] discussed a direct discretization method for the Fredholm equation of the first kind with only one solution and theoretically provided the error estimate by using different discretization methods to solve this kind of Fredholm equation. Hou-de Han et al.[8] studied existence and uniqueness of the solution of the Fredholm equation of the first kind arisen from the Dirichlet problem for two dimensional Laplace's equation. With the introduction of an identification index \( \gamma_n \), it was proved that the necessary and sufficient condition for the existence and uniqueness of the solution of this Fredholm integral equation is \( \gamma_n \neq 0 \).

Numerical Methods for Solving Fredholm Integral Equations of the First Kind.

For Fredholm integral equations of the first kind, the ill-posedness causes the different solvability from other kinds of integral equations. Most of Fredholm integral equations of the first kind, generated in the fields of engineering and image processing, have either infinite solutions or no solution, which makes them extremely difficult to solve. Therefore, it is crucial to study the
Fredholm integral equations of the first kind and obtain the numerical solutions. Research from different aspects has been developed and significant progress has been achieved.

**Regularization Methods**

The regularization method is the most commonly used method for solving the Fredholm integral equations of the first kind, and it is a very effective method to solve the ill-posed problems. In recent years many scholars have utilized various regularization methods to solve the Fredholm integral equations of the first kind and have acquired quite satisfactory results.

M. P. Rajan\(^9\) suggested a modified convergence analysis for solving Fredholm integral equations of the first kind. Using an a priori parameter choice strategy of choosing the regularization parameter, a class of discrepancy principle is proposed under the Hilbert space setting and is illustrated numerically. Enrico De Micheli et al.\(^{10}\) studied the Fredholm equations of the first kind, in the sense of Hadamard. If the integral operator is self-adjoint and admits a set of eigenfunctions, then a formal solution can be written in terms of eigenfunction expansions. One of the possible regularization methods consists in truncating this formal expansion after restricting the class of admissible solutions through a-priori global bounds. Fu-Rong Lin et al.\(^{11}\) presented a new regularization method based on a weighted H1 seminorm. Numerical results showed that the proposed regularization method can restore edges as well as details.

**Wavelet Methods**

Wavelet methods have been used to obtain the numerical solutions of the integral equations since 1991. In recent years, various wavelet functions have been applied to solve the Fredholm integral equations of the first kind, such as CAS wavelets, Legendre wavelets, Coifman wavelets, Haar wavelets, etc., and quite good results have been produced.

S. Yousefi et al.\(^{12}\) utilized the CAS wavelet approximations method to reduce the Fredholm integral equations to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique. Xufeng Shang et al.\(^{13}\) suggested the use of continuous Legendre multi-wavelets constructed on [0, 1] as a basis in Galerkin method, to reduce the solution of Fredholm equations of the first kind to a system of algebraic equations. E. Babolian et al.\(^{14}\) employed wavelets as basis functions in the moment method for solving Fredholm integral equations of the first kind, reducing the order of the linear equation, rather than making the matrix sparse. Hojatollah Adibi et al.\(^{15}\) adopted Chebyshev wavelets, constructed on the unit interval as basis in Galerkin method, to reduce solving the Fredholm integral equations of the first kind to a system of algebraic equations. The properties of Chebyshev wavelets are used to make the wavelet coefficient matrices sparse which eventually leads to the sparsity of the coefficients matrix of obtained system.

**Multilevel Iteration Methods**

The multilevel iteration method have been developed to solve the integral equations since 1990s. Chen, Xu, and Yang\(^{16-17}\) were the first scholars to utilize the multilevel iteration method to solve Fredholm integral equations of the first kind. By combining the Tikhonov regularization and the Multilevel Galerkin method, they truncated the dense matrix to be a sparse matrix so that a fast algorithm can be generated to solve Fredholm equations of the first kind. Fan-chun Li et al.\(^{18-20}\) adopted various multilevel iteration methods to solve Fredholm integral equations of the first kind with disturbed initial data, proposed a fast algorithm with selection of regularized parameters. The complexity and convergence rate of the fast algorithm was analyzed and the approximation solution was proven to be optimized.

**Smooth Methods**

By adding smoothing constraints and adjusting the corresponding parameters, smooth methods can be used to find the steady solutions of Fredholm integral equations of the first kind. De-ming Wang\(^{21}\) considered \(p\) smoothing factors and solved respectively to obtain \(p\) steady solutions with smoothing factors, then the steady solution of the original Fredholm equation can be acquired via
the use of the extrapolation method. Then he worked with Bo Bi to develop a Multi-constraint smooth method for solving Fredholm integral equations of the first kind. A. Kyurkchan et. al. reduced the well-posedness of diffraction problems to Fredholm integral equations of the first kind with a smooth kernel, and utilized the auxiliary source method and the method of extended boundary conditions to solve Fredholm integral equations of the first kind with a smooth kernel. Shaobo Lin et. al. studied the convergence rate for solving Fredholm integral equations of the first kind by using the well-known collocation method. By constructing an approximate interpolation neural network, they deduced the convergence rate of the approximate solution by only using continuous functions as basis functions for the Fredholm integral equations of the first kind. This convergence rate is bounded in terms of a modulus of smoothness.

**Other Methods**

Besides the aforementioned methods for solving Fredholm integral equations of the first kind, other methods have also been developed such as adaptive numerical method, discrete kernel method, optimal homotopy asymptotic method, trust region algorithm, Lagrange polynomial interpolation method, etc.

Nikolay Koshev et al. proposed an adaptive finite element method to solve a linear Fredholm integral equation of the first kind. By deriving a posteriori error estimates in the functional to be minimized and in the regularized solution to this functional, corresponding adaptive algorithms were formulated. Numerical experiments justified the efficiency of the posteriori estimates applied both to the computationally simulated and experimental backscattered data measured in microtomography. Mohammad Almousa et al. presented a semi-analytical method called the optimal homotopy asymptotic method (OHAM) for solving the linear Fredholm integral equations of the first kind. Three examples are discussed to show the ability of the method to solve the linear Fredholm integral equations of the first kind. The results indicated that the method is very effective and simple. Miao Wu implemented the trust region algorithm to solve two kinds of Fredholm integral equations of the first kind, i.e., 1-D with weak singular kernel, 2-D and 3-D with nonsingular kernel. Numerical results were compared with the experimental results and validated the feasibility and effectiveness of the trust region algorithm in this research. Ai-ping Pang developed an slow-solving algorithm to obtain the solutions of Fredholm integral equations of the first kind. With the thought of considering all the possibilities of the unknown function, the data were slowly searched to acquire a more accurate solution of the Fredholm equation of first kind.

**Conclusions and Prospects**

The solving and analyzing of Fredholm integral equations of the first kind have significant meaning for the practical problems in the fields of engineering, physics, image processing, etc. Various numerical methods were presented in this paper for solving Fredholm integral equations of the first kind, including regularization methods, wavelet methods, multilevel iteration methods, etc. Regularization methods utilize different regularizing operators to solve different Fredholm integral equations. Wavelet methods employ various wavelet functions as the basis functions to approach Fredholm integral equations of the first kind such that the integral equations can be solved by improving the basis functions. Multilevel iteration methods truncate dense matrices to be sparse matrices so that fast algorithms can be generated to solve Fredholm equations of the first kind. Other methods include adaptive numerical method, discrete kernel method, optimal homotopy asymptotic method, trust region algorithm, Lagrange polynomial interpolation method, and etc. All above methods attempt to approach the Fredholm integral equations from different aspects. Nevertheless, it is very difficult in not only choosing the approaching functions, but obtaining the numerical solutions of the approaching functions.

It is desired to provide a simple discretizing formula such that Fredholm integral equations can be discretized to be a system of linear algebraic equations. However, most of the iteration algorithms can’t yield desirable results for solving of large-scale linear equations. Intelligence optimization algorithms, such as genetic algorithms, neural network algorithms, particle swarm algorithms, can
be good candidates for solving Fredholm integral equations of the first kind. Intelligence algorithms are constructed by mimicking natural phenomena and real engineering problems then comprehensively utilizing the knowledge of physics, biological evolution, artificial intelligence, neural science, etc. Intelligence algorithms can be used to solve engineering and scientific problems due to their quickness, randomness, and high efficiency. Desirable results could be obtained when intelligence algorithms are used to solve Fredholm integral equations of the first kind after being discretized to be linear algebraic equations.

References
[8] Hou-de Han et. al., Necessary and sufficient conditions for the sole existence of the solution for a Fredholm integral equation of the first kind, Scientia Sinica (Mathematica), 2015, No.8, pp. 1231-1248.


