Fixed Point Theorems for Generalized $\Psi\Phi$-Khan-contractions in $b$-rectangular Metric Spaces

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Abstract. In this paper, we introduce the concept of generalized $\Psi\Phi$-Khan contraction in $b$-rectangular metric spaces. By using the method of the fixed point theory, we obtain some fixed point theorems for such mapping in complete $b$-rectangular metric space.

Introduction and Preliminaries

The contraction principle introduced by Banach [1] is one of the most important results in mathematical analysis. Indeed, it is widely used as the source of metric fixed point theory. In 2009, D. Doric [2] introduced a new concept of contraction called generalized ($\Psi\Phi$)-weak contraction and proved a common fixed point theorem which generalized the Banach contraction principle. Afterward, some authors have obtained fixed point theorems for some kinds of ($\Psi\Phi$)-weak contractive mappings (see [3,4,5,8]). In 2016, H. Piri et al. [9] proved Khan type fixed point theorems in a generalized metric space. In [10], they also studied some results of existence and uniqueness of fixed points for a class of mappings satisfying an inequality of rational expressions. Hossein Piri et al. [11] introduced the concept of F-Khan-contractions and proved a fixed point theorem in complete metric spaces. Moreover, Ansari et al. [12] proved C-class function on Khan type common fixed point theorems in generalized metric space.

In 2015, George et al. [13] introduced the concept of rectangular b-metric space and proved an analogue of Banach contraction principle and Kannan's fixed point theorem in this space. Since then many fixed point theorems for various contractions on rectangular b-metric spaces (see [6,14,15]).

In this paper, we will introduce the generalized $\Psi\Phi$-Khan-contraction which satisfy different conditions in $b$-rectangular metric space and then we shall prove some fixed point theorems for these contractions in a partially ordered complete $b$-rectangular metric space.

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers.

Definition 1.1. ([13]) Let $X$ be a nonempty set, $s\geq1$ be a given real number and let $d: X \times X \to [0, \infty]$ be a mapping such that for all $x, y \in X$ and distinct points $u, v \in X$, each distinct from $x$ and $y$:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) = s[d(x, u) + d(u, v) + d(v, y)]$ ($b$-rectangular inequality).

Then $(X, d)$ is called a $b$-rectangular metric space or a $b$-generalized metric space ($b$-g.m.s.).

In 1984, Khan et al. [7] introduced altering distance functions as follows:

Definition 1.2. ([7]) A function $\Psi: [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(a) $\Psi$ is non-decreasing and continuous;
(b) $\Psi(t) = 0$ if and only if $t = 0$.

The family of all altering distance functions is denoted by $\Psi$.

Definition 1.3. We denote by $\varphi$ the set of function $\varphi: [0, \infty) \to [0, \infty]$ satisfying the following hypotheses:
(a) $\phi$ is lower semi-continuous;
(b) $\phi(t) = 0$ for all $t = 0$.

The following lemma will be used for proving our main results.

**Lemma 1.** ([6]) Let $(X, d)$ be a $b$-rectangular metric space and let $\{x_n\}$ be a Cauchy sequence in $X$ such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ can converge to at most one point.

**Lemma 2.** ([6]) Let $(X, d)$ be a $b$-g.m.s. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in $X$ are such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, with $x \neq y$, and $x_n \neq y_n$ for $n \in \mathbb{N}$. Then we have

$$\frac{1}{s} d(x, y) \leq \lim \inf_{n \to \infty} d(x_n, y_n) \leq \lim \sup_{n \to \infty} d(x_n, y_n) \leq sd(x, y).$$

If $y \in X$ and $\{x_n\}$ is a Cauchy sequence in $X$ with $x_n \neq x_m$ for infinitely many $m, n \in \mathbb{N}$ with $n \neq m$, converging to $x \neq y$, then

$$\frac{1}{s} d(x, y) \leq \lim \inf_{n \to \infty} d(x_n, y) \leq \lim \sup_{n \to \infty} d(x_n, y) \leq sd(x, y), \text{ for all } x \in X.$$

**Main Results**

In this section, we introduce the concept of generalized $\Psi$-$\phi$-$Khan$-contraction and prove the fixed point theorems for such mapping.

**Definition 2.1.** Let $(X, d)$ be a $b$-rectangular metric space with $s > 1$, and let $T : X \to X$ be a self mapping. A mapping $T$ is said to be a generalized $\Psi$-$\phi$-$Khan$-contraction, if there exists $\psi : \Phi \to [0, \infty)$ such that for all distinct $x, y \in X$, the following conditions hold:

$$\psi(s^2 d(Tx, Ty)) \leq \begin{cases} \psi(M(x, y)) - \phi(M(x, y)), & \text{if } \max\{d(x, Ty), d(Tx, y)\} \neq 0, \\ \psi(d(x, y)), & \text{if } \max\{d(x, Ty), d(Tx, y)\} = 0, \end{cases}$$

where $M(x, y) = \max\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{\max\{d(x, Tx), d(Tx, y)\}}\}$.

**Theorem 2.1.** Let $(X, \preceq, d)$ be a partially ordered complete $b$-rectangular metric space with parameter $s > 1$. Let $T : X \to X$ be a non-decreasing mapping with respect to $\preceq$ such that there exists $x_0 \in X$ with $x_0 \preceq T x_0$. Suppose that following conditions are satisfied:

(i) $T$ is a generalized $\Psi$-$\phi$-$Khan$-contraction;

(ii) $T$ is continuous.

Then $T$ has a fixed point. Moreover, the set of fixed points $T$ is well ordered only if $T$ has one and only one fixed point.

**Proof.** Let $x_0 \in X$ such that $x_0 \preceq T x_0$ and $\{x_n\}$ be a sequence in $X$ with $x_{n+1} = T x_n = T^n x_0$ for all $n \in N \cup \{0\}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = T x_n$ is a fixed point of $T$. Therefore, we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since $x_0 \preceq T x_0 = x_1$ and $T$ is non-decreasing, we have $x_1 = T x_0 \preceq T x_1 = x_2$ Continuing this process, we obtain that $x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$ for all $n \in \mathbb{N}$.

We shall divide the proof into two cases.

**Case 1.** Assume that $\max\{d(x_n, Tx_m), d(Tx_n, x_m)\} \neq 0$ for all $m \in \mathbb{N}$ and $n \in N \cup \{0\}$. Then from (2.1) we have

$$\psi(s^2 d(x_n, x_{n+1})) = \psi(s^2 d(Tx_{n-1}, Tx_n)) \leq \psi(M(x_{n-1}, x_n)) - \phi(M(x_{n-1}, x_n)), \text{ where}$$

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_n) + d(x_n, Tx_n)d(x_n, Tx_{n-1})}{\max\{d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\}}\}$$

$$= d(x_{n-1}, x_n).$$

Assume that $d(x_{n-1}, x_n) = 0$, for all $n \in N \cup \{0\}$. From (2.2), (2.3) and the definition of $\psi$ we have $d(x_n, x_{n+1}) = 0$, which is a contradiction. So $d(x_{n-1}, x_n) \neq 0$, using (2.2) and (2.3) we have
\[
\psi(s^2d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n)) < \psi(d(x_{n-1}, x_n)).
\]  
(2.4)

Since \(\psi\) is non-decreasing, so from (2.4), implies that 
\[
d(x_n, x_{n+1}) \leq \frac{1}{s^2}d(x_{n-1}, x_n) < d(x_{n-1}, x_n),
\]  
(2.5)

for all \(n \in \mathbb{N}\).

It follows that the sequence \(\{d(x_n, x_{n+1})\}\) is non-increasing. Hence, there exists \(r > 0\) such that 
\[
limit_{n \to \infty} d(x_n, x_{n+1}) = r.
\]

Then, from (2.4), we have \(\psi(s^2r) \leq \psi(r) - \varphi(r) < \psi(r)\), which is a contradiction. We have 
\[
r = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0.
\]  
(2.6)

Assume first that \(x_n = x_m\) for some \(n > m\). Similarly as in the proof of Theorem 1 of [6], we get a contradiction. So we may assume that \(x_n \neq x_m\) for all \(n, m \in \mathbb{N}\) with \(n \neq m\).

Analogously, we can prove that \(\lim_{n \to \infty} d(x_n, x_{n+2}) = 0\), by substituting \(x = x_{n-1}\) and \(y = x_{n+1}\) in (2.1), we obtain 
\[
\psi(s^2d(x_n, x_{n+2})) \leq \psi(M(x_{n-1}, x_{n+1})) - \varphi(M(x_{n-1}, x_{n+1})),
\]  
(2.7)

where 
\[
M(x_{n-1}, x_{n+1}) = \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_{n+1})}{\max\{d(x_{n-1}, x_{n+1}), d(x_n, x_{n+1})\}}\} 
\leq \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_{n+1})}{d(x_n, x_{n+1})}\} 
\leq \max\{sd(x_{n-1}, x_n) + sd(x_{n+1}, x_{n+2}), sd(x_{n+1}, x_{n+2}) + sd(x_{n-1}, x_n)\} 
= sd(x_{n-1}, x_n) + sd(x_{n+1}, x_{n+2}).
\]  
(2.8)

Using (2.7) and (2.8) we have 
\[
\psi(s^2d(x_n, x_{n+2})) \leq \psi(M(x_{n-1}, x_{n+1})) - \varphi(M(x_{n-1}, x_{n+1}))
\leq \psi(sd(x_{n-1}, x_n) + sd(x_{n+1}, x_{n+2}) + sd(x_{n+2}, x_{n+1})).
\]  
(2.9)

Since \(\psi\) is non-decreasing and \(s > 1\), then it is easy to see from (2.9), we get 
\[
d(x_n, x_{n+2}) \leq \frac{1}{s^2}d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2}).
\]

Taking limit \(n \to \infty\) on both sides of above inequality and use (2.6), we have 
\[
limit_{n \to \infty} d(x_n, x_{n+2}) = 0.
\]

Next, we will show that \(\{x_n\}\) is a \(b\)-g.m.s Cauchy sequence in \(X\). We will prove by using a contradiction technique. Assume that \(\{x_n\}\) is not a Cauchy sequence. Therefore there exists \(\varepsilon > 0\) for we can find two subsequences \(\{x_{m_i}\}\) and \(\{x_{n_i}\}\) of \(\{x_n\}\) such that \(n_i > m_i \geq i, n_i > m_i + 2\) and \(d(x_{m_i}, x_{n_i}) \geq \varepsilon\). Then, 
\[
d(x_{m_i}, x_{n_i-2}) < \varepsilon.
\]  
(2.10)

Taking the upper limit in (2.10) as \(i \to \infty\), we get 
\[
\limsup_{i \to \infty} d(x_{m_i}, x_{n_i-2}) \leq \varepsilon.
\]  
(2.11)

Using \(b\)-rectangular inequality, we have 
\[
d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_{i+1}}) + sd(x_{m_{i+1}}, x_{n_{i-1}}) + sd(x_{n_{i-1}}, x_{n_i}).
\]

Taking the upper limit in the above inequality and using (2.6), we get
In addition, we have \( \frac{\varepsilon}{s} \leq \limsup_{i \to \infty} d(x_{m_i+1}, x_{n_i-1}). \) \hspace{1cm} (2.12)

By taking the upper limit in the above inequality and using (2.6), we get \( \frac{\varepsilon}{s} \leq \limsup_{i \to \infty} d(x_{m_i}, x_{n_i-2}). \) \hspace{1cm} (2.13)

Now putting \( x=x_m \) and \( y=x_{n-2} \) in (2.1), then we obtain

\[
\psi\left(s^2d(x_{m_i+1}, x_{n_i-1})\right) = \psi\left(s^2d(Tx_{m_i}, Tx_{n_i-2})\right)
\leq \psi(M(x_m, x_{n_i-2})) - \varphi(M(x_m, x_{n_i-2})),
\]

where

\[
M(x_m, x_{n_i-2}) = \max\left\{d(x_{m_i}, x_{n_i-2}), \frac{d(x_{m_i}, x_{m_i+1})d(x_{m_i}, x_{n_i-1}) + d(x_{n_i-2}, x_{n_i-1})d(x_{n_i-2}, x_{m_i+1})}{\max\{d(x_{m_i}, x_{n_i-1}), d(x_{m_i+1}, x_{n_i-2})\}}\right\}
\leq \max\{d(x_{m_i}, x_{n_i-2}), d(x_{m_i}, x_{m_i+1}) + d(x_{n_i-2}, x_{m_i-1})\}.
\]

Taking the upper limit \( i \to \infty \) in (2.15), by using (2.6) and (2.11) we get

\[
\limsup_{i \to \infty} M(x_{m_i}, x_{n_i-2}) \leq \varepsilon.
\]

In the same way, by taking the lower limit as \( i \to \infty \) in (2.15) and using (2.6), we get

\[
\frac{\varepsilon}{s} \leq \liminf_{i \to \infty} M(x_{m_i}, x_{n_i-2}).
\]

Next, taking the upper limit as \( i \to \infty \) in (2.14) and using (2.12), (2.16) and (2.17) we have

\[
\psi\left(s^2 \cdot \frac{\varepsilon}{s}\right) \leq \psi\left(s^2 \limsup_{n \to \infty} d(x_{m_i+1}, x_{n_i-1})\right)
\leq \psi\left(\limsup_{n \to \infty} M(x_m, x_{n_i-2})\right) - \varphi\left(\liminf_{n \to \infty} M(x_m, x_{n_i-2})\right)
\leq \psi(\varepsilon) - \varphi\left(\liminf_{n \to \infty} M(x_m, x_{n_i-2})\right)
\leq \psi(\varepsilon).
\]

Since \( \psi \) is non-decreasing function, we have \( s \leq \varepsilon \), which is a contradiction. Hence \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( (X,d) \) is complete, there exists \( u \in X \) such that \( \lim_{n \to \infty} d(x_n, u) = 0 \). We show that \( u \) is a fixed point of \( T \). Since \( T \) is continuous, then

\[
\lim_{n \to \infty} d(Tx_n, Tu) = \lim_{n \to \infty} d(x_{n+1}, Tu) = 0.
\]

Thus by Lemma 1, it follows that \( x_n \) differs from both \( u \) and \( Tu \) for \( n \) sufficiently large. Using the \( b \)-rectangular inequality, we obtain

\[
d(u, Tu) \leq s[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)].
\]

Taking limit as \( n \to \infty \), we have

\[
d(u, Tu) \leq \lim_{n \to \infty} s[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] = 0.
\]

So we have \( Tu = u \). Then, \( u \) is a fixed point of \( T \). Finally, suppose that the set of fixed points of \( T \) is well ordered. Assume, on the contrary, that \( u, v \) are two fixed points of \( T \) such that \( u \neq v \). Hence, from (2.1) with \( x=u \) and \( y=v \) we have

\[
\psi(s^2d(Tu, Tv)) \leq \psi(M(u, v)) - \varphi(M(u, v)),
\]

where

\[
M(u, v) = \max\{d(u, v), \frac{d(u, Tu)+d(v, Tu)d(v, Tv)}{\max\{d(u, Tu)+d(Tu, v)\}}\} = d(u, v).
\]
By (2.18) and (2.19), we get \[\psi(s^2d(u, v)) \leq \psi(d(u, v)) - \varphi(d(u, v)) < \psi(d(u, v)).\] (2.20)

By virtue of (2.20) and the monotonicity of \(\psi\), it follows that \(s^2d(u, v) \leq d(u, v)\).

Therefore, \(d(u, v) = 0\), i.e. \(u=v\). Then \(T\) has a unique fixed point. Conversely, suppose that \(T\) has a unique fixed point, then the set of fixed points of \(T\) is singleton. Hence, it is well ordered.

Case 2. Assume that there exists \(m \in \mathbb{N}\) such that \(\max\{d(x_{m-1}, T x_{m}), d(T x_{m-1}, x_m)\} = 0\).

By condition (2.1), it follows that \(d(T x_{m-1}, T x_m) = 0\) and hence \(x_m = T x_m\). This completes the existence of a fixed point of \(T\). The uniqueness follows as in Case 1.

Theorem 2.2 Under the hypotheses of Theorem 2.1, if the continuity of \(T\) is replaced by the condition that \(x_n \preceq x\), for all \(n \in \mathbb{N}\) whenever \(\{x_n\}\) is a non-decreasing sequence in \(X\) such that \(x_n \to x \in X\),

Then \(T\) has a fixed point in \(X\).

Proof. Following similar arguments to those given in the proof of Theorem 2.1, we construct an increasing sequence \(\{x_n\}\) in \(X\) such that \(x_n \to u\) for some \(u \in X\). Using the assumption on \(X\), we have that \(x_n \preceq u\) for all \(n \in \mathbb{N}\). Now, we show that \(Tu = u\). By (2.1), we have

\[
\psi(s^2d(Tx_n, Tu)) \leq \psi(M(x_n, u)) - \varphi(M(x_n, u)),
\]

where

\[
M(x_n, u) = \max\{d(x_n, u), \frac{d(x_n, Tx_n)d(x_n, Tu) + d(u, Tu)d(u, Tx_n)}{\max\{d(x_n, Tu), d(Tx_n, u)\}}\}.
\]

Letting \(n \to \infty\) in (2.22), we obtain that \(M(x_n, u) \to d(u, Tu)\).

Next, we take the upper limit as \(n \to \infty\) in (2.21) and use Lemma 2 and (2.22) we get

\[
\psi(s \limsup_{n \to \infty} d(Tx_n, Tu)) \leq \psi(s \cdot \liminf_{n \to \infty} d(Tx_n, Tu))
\]

\[
\leq \psi(\limsup_{n \to \infty} M(x_n, u)) - \varphi(\liminf_{n \to \infty} M(x_n, u))
\]

\[
= \psi(d(u, Tu)) - \varphi(\liminf_{n \to \infty} M(x_n, u))
\]

\[
< \psi(d(u, Tu)).
\]

By virtue of (2.23) and monotonicity of \(\psi\), it follows that \(s \cdot d(u, Tu) \leq d(u, Tu)\), which is a contradiction. Therefore, we get \(u = Tu\) and then \(u\) is a fixed point of \(T\).

References


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