The Calculation Methods for Solving a Kind of Boundary Value Problems in Elasticity

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Keywords: Nonlinear boundary value condition, Elasticity, extrapolation algorithm, Mechanical quadrature methods, Boundary integral equations.

Abstract. This paper will study the numerical solutions for equations with a kind of boundary value conditions. The equations will be converted into nonlinear boundary integral equations by the potential theory, in which logarithmic singularity and Cauchy singularity are calculated simultaneously. Mechanical quadrature methods(MQMs) are presented to solve the nonlinear equations that the accuracy of the solutions are three order. According to the asymptotical compact convergence theory, the errors with an odd powers asymptotic expansion is obtained. Some results are shown regarding these approximations for problems by the numerical example.

Introduction

This paper will describe the isolated elastic equations on a bounded planar regions $\Omega$ in the plane with nonlinear boundary value conditions:

$$
\begin{align*}
\sigma_{ij,j} &= 0, \quad \text{in } \Omega \\
\mathbf{p} &= -\mathbf{g}(x, \eta) + \mathbf{f}(x), \text{ on } \Gamma, i, j = 1, 2,
\end{align*}
$$

(1)

Where $\Omega \subset R^2$ is a connected domain with a smooth closed curve $\Gamma$, the stress tensors are $\sigma_{ij}$, $n = (n_1, n_2)$ is the unit outward normal vector on $\Gamma$, the tractor vector is assumed given $\mathbf{p} = (p_1, p_2)\hat{\mathbf{r}}$ with $\mathbf{p} = (\sigma_{i1} n_1 + \sigma_{i2} n_2)$, $\mathbf{f}(x) = (f_1(x), f_2(x))\hat{\mathbf{r}}$ and continuous on $\Gamma$, and $\mathbf{g}(x, \eta) = (\mathbf{g}_1(x, \eta), \mathbf{g}_2(x, \eta))\hat{\mathbf{r}}$ that $\mathbf{g}_i(x, \eta)$ is a nonlinear function corresponding to $\eta\hat{\mathbf{r}}$. Following vector computational rules, the repeated subscripts imply the summation from 1 to 2.

The problem are studied in many applications, e.g. the circular ring problems and the instance problems[14; 18], and the bending of prismatic bars[22]. Some methods have been proposed for solving the elasticity. The nonconforming mixed finite element methods are established by Hu and Shi[12] to solve the elasticity with a linear boundary. The least-square methods are introduced by Cai et al.[4] for obtaining the solution of the elastic problems. Chen and Hong[5] solved the hyper singular integral equations applying the dual boundary element methods in elasticity. Kuo et al.[15] solve true and spurious eigensolutions of the circular cavity problems used dual methods. Li and Nie[16] researched the stressed axi-symmetric rods problems by a high-order integration factor method. And the mechanical quadrature methods(MQMs) are adopted by Cheng et al.[6] to solve Steklov eigensolutions in elasticity to obtain high accuracy solutions. The Eqs.(1) are converted into the boundary integral equations[3; 6; 21] (BIEs) with the Cauchy and loga-rithmic singularities by the variational method.

$$
\frac{\sigma_{ij}}{2} \eta_j(y) + \int_{\Gamma} k^*_i(y, x) \mathbf{f}(x) dx = \int_{\Gamma} h^*_i(y, x) \mathbf{p}(x) dx, \quad y = (y_1, y_2) \in \Gamma,
$$

(2)

Where $\delta_{ij}$ is the Kronecker delta for $i, j = 1, 2$, and the kernels:
\[
\begin{aligned}
  h^*_y &= \frac{1}{8\pi\mu(1-\nu)} \{ -(3-4\nu)\sigma_{ij} \ln r+y_i r_j \}, \\
  k^*_y &= \frac{1}{4\pi(1-\nu)} \frac{\partial}{\partial n} \{ (1-2\nu)\sigma_{ij} + 2r_i r_j \} + (1-2\nu)(n_i r_j - n_j r_i) \}.
\end{aligned}
\]

The kernels are Kelvin's fundamental solutions[3]. The Poisson ratio is \( \nu = \lambda/[2(\lambda + \mu)] \), \( r_j \) means the derivative to \( x_j \), and \( r = \sqrt{(y_1-x_1)^2 + (y_2-x_2)^2} \) is the distance of \( x \) and \( y \). The parts \( \int_{\Gamma} k^*_y(y,x)\overline{\eta}_j(x)ds_x \) of the Eqs.(2) are the Cauchy singularity and the parts \( \int_{\Gamma} h^*_y(y,x)\overline{\eta}_j(x)ds_x \) are the logarithmic singularity.

The nonlinear integral equations are obtained after the boundary conditions are substituted into Eqs.(2):

\[
\frac{\sigma_y}{2} \overline{\eta}_j(y) + \int_{\Gamma} k^*_y(y,x)\overline{\eta}_j(x)ds_x = \int_{\Gamma} h^*_y(y,x)\overline{\eta}_j(x)ds_x + \int_{\Gamma} h^*_y(y,x)\overline{f}_j(x)ds_x.
\]


Since this is a nonlinear system including the logarithmic singularity and the Cauchy singularity, the difficulty is to obtain the discrete equations appropriately. The displacement vector and stress tensor in \( \Omega \) can be calculated[8; 11].

Richardson extrapolation algorithms(EAs) are pretty effective parallel algorithm which are based on asymptotic expansion about errors. The solutions, which are solved on coarse grid and fine grid, are used to construct high accuracy solutions. Its’ notified as good stability and optimal computational complexity. Cheng et al.[7] harnessed extrapolation algorithms to obtain high accuracy order for Steklov eigenvalue of Laplace equations. Huang and Lü established extrapolation algorithms to obtain high accuracy solutions for solving Laplace equations on arcs[13] and Steklov eigenvalue problem in elasticity[10].

**Mechanical Quadrature Methods**

We firstly Define the notation of the integral operators on boundary \( \Gamma \) as follows to simplify the equations:

\[
\begin{aligned}
  (K_{ij}\eta)(y) &= \int_{\Gamma} k^*_{ij}(y,x)\eta(x)ds_x \quad y \in \Gamma, i,j = 1,2, \\
  (H_{ij}\eta)(y) &= \int_{\Gamma} h^*_{ij}(y,x)\eta(x)ds_x \quad y \in \Gamma, i,j = 1,2.
\end{aligned}
\]

So Eqs.(2) can be simplified as the following operator equations:

\[
\begin{bmatrix}
  \frac{1}{2}I_0 + K_{11} & \frac{1}{2}I_0 + K_{12} \\
  K_{21} & \frac{1}{2}I_0 + K_{22}
\end{bmatrix}
\begin{bmatrix}
  \overline{\eta}_1 \\
  \overline{\eta}_2
\end{bmatrix}
= \begin{bmatrix}
  H_{11} & H_{12} \\
  H_{21} & H_{22}
\end{bmatrix}
\begin{bmatrix}
  \overline{g}_1 + \overline{f}_1 \\
  \overline{g}_2 + \overline{f}_2
\end{bmatrix},
\]

where \( I_0 \) is an identity operator.
Suppose \( C_{2m}^2[0, 2\pi] \) be the set of \( 2m \) times differentiable periodic functions in which the periodic of all functions are \( 2\pi \). A regular parameter mapping: \([0, 2\pi] \to \Gamma\) is introduced, so the smooth closed curve \((x_1(s), x_2(s))\) satisfying \(|x'(s)|^2 = |x'_1(s)|^2 + |x'_2(s)|^2 > 0\) with \( x(s) \in C_{2m}^2[0, 2\pi], i = 1, 2\).

The singularity of the kernels \( h^*_j, k^*_j \) will be analyzed as \( t \to s \) before the application of discrete methods.

Since the expression.

\[
\frac{n_{ij} r - n_{ji} r}{r} = (-1)^i \frac{1 + o(t-s)}{(t-s) + o(t-s)}, \quad i \neq j,
\]

so the Cauchy singularity of \( k^*_j \) comes from the part \( (n_{ij} r - n_{ji} r)/r \). Moreover, the logarithmic singularity of \( h^*_j \) comes from the component \( \log|x-y| \) and the other parts of the operators are smooth. So the operators \( H_{i,j} \) and \( K_{i,j} \) will be divided into several parts.

The logarithmic singular operator \( H_{i,j} \) will be divided into three parts on \( C_{2m}^2[0, 2\pi] \) as follows:

\[
(A_0 \eta)(t) = \int_0^{2\pi} a_0(t, \tau) \eta(\tau) |x'(\tau)| d\tau,
\]

with \( a_0(t, \tau) = \bar{c}_0 \ln |2e^{-1/2} \sin \frac{\tau}{2}| \), \( \bar{c}_0 = -(3-4\nu)/[8\pi \mu(1-\nu)] \), and

\[
(B_0 \eta)(t) = \int_0^{2\pi} b_0(t, \tau) \eta(\tau) |x'(\tau)| d\tau,
\]

with \( b_0(t, \tau) = \bar{c}_0 \ln |x(t) - x(\tau)| \ln |2e^{-1/2} \sin \frac{\tau}{2}| \), and

\[
(B_0 \eta)(t) = \int_0^{2\pi} b_0(t, \tau) \eta(\tau) |x'(\tau)| d\tau,
\]

with \( b_0(t, \tau) = c_i r^2 r_j, c_i = 1/[8\pi \mu(1-\nu)] \).

The Cauchy singular operator \( K_{i,j} \) will also be divided into three parts on \( C_{2m}^2[0, 2\pi] \) as follows:

\[
(C_0 \eta)(t) = \int_0^{2\pi} c_0(t, \tau) \eta(\tau) |x'(\tau)| d\tau,
\]

with \( c_0(t, \tau) = c_2 (n_{11} r_2 - n_{12} r_1)/r, c_2 = -(1-2\nu)/[4\pi(1-\nu)] \), and

\[
(M_{ii} \eta)(t) = \int_0^{2\pi} m_{ii}(t, \tau) \eta(\tau) |x'(\tau)| d\tau, \quad i = 1, 2,
\]

with \( m_{ii}(t, \tau) = c_3 \left(c_5 \tau + (1-2\nu) + 2r_i r_j \right)/r, c_3 = -1/[4\pi(1-\nu)] \), and

\[
(M_{ij} \eta)(t) = \int_0^{2\pi} m_{ij}(t, \tau) \eta(\tau) |x'(\tau)| d\tau, \quad i = 1, 2, i \neq j,
\]

with \( m_{ij}(t, \tau) = c_3 \left(c_5 \tau + 2r_i r_j \right)/r \).

We can find that \( B_0, B_{i,j}, M_{i,j} \) are smooth operators, \( A_0 \) is a logarithmic singular operator, and \( C_0 \) is a Cauchy singular operator.

Then the equivalent equation of Eqs.(4) will be shown as:

\[
\left(\frac{1}{2} I + C + M \right) \eta = (A + B) g(\eta) + f,
\]

(7)
where \( \eta(t) = (\eta_1(x(t)), \eta_2(x(t))) \), \( f(t) = (f_1, f_2)^T \) with \( f_i = H_{i,j} \tilde{f}_j(x(t)), g(\eta(t)) = \tilde{g}(x(t), \eta(t)) \) and

\[
I = \begin{pmatrix} I_0 & 0 \\ 0 & I_0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & C_0 \\ C_0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix},
\]

\[
B = \begin{pmatrix} b_{0} + b_{1} & b_{0} \\ b_{0} + b_{2} & b_{0} + b_{2} \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},
\]
suppose the boundary \( \Gamma \) will be divided into \( 2n, (n \in \mathbb{N}) \) equal parts, so the mesh width will be \( h = \pi / n \) and the nodes will be \( t_j = \tau_j = jh, (j = 0, 1, ..., 2n - 1) \). The smooth integral operators with the period \( 2\pi \) can be easily to obtain high accuracy Nyström's approximations[23]. For example, the approximation operator \( B_0^h \) of \( B_0 \) can be approximated as follows:

\[
B_0^h \eta = h \sum_{j=0}^{2n-1} b_0(t, \tau_j) \eta(\tau_j).
\] (8)

The Nyström's approximation \( B_y^h \) of \( B_y \) and \( M_y^h \) of \( M_y \) can be approximated similarly.

In order to approximate the logarithmic singular operator \( A_0 \), the continuous approximation kernel \( a_p(t, \tau) \) can be Defined as follows:

\[
a_p(t, \tau) = \begin{cases} a_0(t, \tau), & \text{for } |\tau-t| < h, \\
\tau_0 h \ln \left| \frac{\tau - t}{\pi h} \right|, & \text{for } |\tau-t| < h.
\end{cases}
\] (9)

By Sidi's quadrature rules[22], the approximation operator can be constructed as follows:

\[
(A_0^h \eta)(t) = h \sum_{j=0}^{2n-1} a_p(t, \tau_j) \eta(\tau_j) \left| x'(\tau_j) \right|.
\] (10)

The approximation operator can be obtained for the Cauchy singular operator \( C_0, C_0^h \) can be defined as:

\[
(C_0^h \eta)(t_i) = 2c_x a_1(t_i, t_j) h \sum_{j=0}^{2n-1} \cot((t_j - t_i) / 2) \eta(t_j) \left| x'(t_j) \right| \mathcal{E}_{ij},
\] (11)

where

\[
\mathcal{E}_{ij} = \begin{cases} 1, & \text{if } |i-j| \text{ is odd number}, \\
0, & \text{if } |i-j| \text{ is even number}.
\end{cases}
\]

Thus the Eqs.(7) can be rewrite as follows:

\[
\left( \frac{1}{2} I + C^h + M^h \right) \eta_h - (A^h + B^h) g(\eta_h) + f_h,
\] (12)

where \( A^h, B^h, C^h \) and \( M^h \) are the approximate matrixes corresponding to the operators \( A, B, C \) and \( M \), respectively.

So we derive the asymptotic expansion of errors for the solution, and construct the extrapolation algorithm to obtain higher accuracy order solutions.

**Theorem 2.** Consider the asymptotic property and \( x(t), f(t) \in C^2[0, 2\pi] \), there exists a function \( \phi_h \in C^{2m-2}[0, 2\pi] \) independent of \( h \), such that
(\eta - \eta_h) \bigg|_{t_1} = h^3 \partial_1 \bigg|_{t_1} + o(h^3) \tag{13}

An asymptotic expansion about the errors in Eq. (22) implies that the extrapolation algorithms[9] can be applied to the solution of Eqs. (2) to improve the approximate order. The high accuracy order \( o(h^3) \) can be obtained by computing some coarse grids and fine grids on \( \Gamma \) in parallel. The EAs are described as follows:

Use the values at coarse grids and fine grids to calculate the approximate values at \( t_1 \).

\begin{align*}
\eta^*_h(t_1) = \frac{1}{7}(8\eta_{h/2}(t_1) - \eta_h(t_1)).
\end{align*} \tag{14}

**Numerical Example**

**Example 1:** We firstly introduce some denote for \( i = 1, 2 \): \( e^h_i(P) = |\eta_{ih}(P) - \eta_1(P)| \) is the error of the displacement; \( r^h_i(P) = e^h_i(P) / e^{h/2}(P) \) is the error ratio; \( e^*_h(P) = |\eta^*_{ih}(P) - \eta^*_1(P)| \) is the error after one-step EAs; and \( p^h_i(P) = \frac{1}{2}|\eta_{ih/2}(P) - \eta_{ih}(P)| \) is a posteriori error estimate.

Suppose \( \Omega \) is an isotropic elliptical body with the axis \( a = 0.3, b = 0.5 \) in the plane domain. The parameter formulae for the boundary \( \Gamma \) will be described as \( x = 0.3 \cos(t), y = 0.5 \sin(t), t \in [0.2\pi] \).

\begin{table}[h]
\centering
\begin{tabular}{c|cccccc}
\hline
n & 16 & 32 & 64 & 128 & 256 & 512 \\
\hline
\( e^h_1(P_1) \) & 8.466E-4 & 1.040E-4 & 1.296E-5 & 1.618E-6 & 2.022E-7 & 2.527E-8 \\
\( r^h_1(P_1) \) & 8.137 & 8.028 & 8.011 & 8.002 & 8.000 & 8.000 \\
\( e^*_h(P_1) \) & 4.12E-07 & 1.29E-08 & 4.03E-10 & 1.24E-11 & 3.93E-13 & \\
\( \pi^*_h(P_1) \) & 31.92 & 31.99 & 32.41 & 31.68 & & \\
\hline
\( e^h_2(P_2) \) & 6.085E-4 & 7.501E-5 & 9.292E-6 & 1.160E-6 & 1.450E-7 & 1.813E-8 \\
\( r^h_2(P_2) \) & 8.112 & 8.073 & 8.008 & 8.000 & 8.000 & 8.000 \\
\( e^*_h(P_2) \) & 3.91E-07 & 1.24E-08 & 4.00E-10 & 1.30E-11 & 4.03E-13 & \\
\( \pi^*_h(P_2) \) & 31.47 & 31.09 & 30.85 & 32.16 & & \\
\hline
\end{tabular}
\caption{The errors, errors ratio of \( \eta_{ih}(P) \) at points \( P = P_1, P_2 \).}
\end{table}

We firstly calculate the numerical solutions \( \eta_h = (\eta_{ih}, \eta_{2ih})^T \) on the boundary \( \Gamma \) following Eqs.(16). Table 1 lists the approximate values of \( \eta_{ih}(p) \) at points \( P_1 = (a \cos \frac{\pi}{3}, b \sin \frac{\pi}{3}) \) and \( P_2 = (a \cos \frac{\pi}{6}, b \sin \frac{\pi}{6}) \). Table 2 lists the approximate values of \( \eta_{2ih}(p) \) at points \( P_1 = (a \cos \frac{\pi}{4}, b \sin \frac{\pi}{4}) \) and \( P_2 = (a \cos \frac{\pi}{4}, b \sin \frac{\pi}{4}) \).

\begin{table}[h]
\centering
\begin{tabular}{c|cccccc}
\hline
n & 16 & 32 & 64 & 128 & 256 & 512 \\
\hline
\( e^h_1(P_1) \) & 4.169E-4 & 5.100E-5 & 6.309E-6 & 7.805E-7 & 9.753E-8 & 1.219E-8 \\
\( r^h_1(P_1) \) & 8.175 & 8.083 & 8.018 & 8.003 & 8.000 & 8.000 \\
\( e^*_h(P_1) \) & 7.67E-06 & 2.44E-07 & 7.80E-09 & 2.45E-10 & 7.75E-12 & \\
\( \pi^*_h(P_1) \) & 31.44 & 31.29 & 31.77 & 31.65 & & \\
\hline
\( e^h_2(P_2) \) & 9.338E-4 & 1.147E-4 & 1.420E-5 & 1.774E-6 & 2.217E-7 & 2.771E-8 \\
\( r^h_2(P_2) \) & 8.138 & 8.080 & 8.005 & 8.001 & 8.000 & 8.000 \\
\( e^*_h(P_2) \) & 8.30E-06 & 2.75E-7 & 8.75E-9 & 2.79E-10 & 8.83E-12 & \\
\( \pi^*_h(P_2) \) & 30.21 & 31.40 & 30.69 & 31.56 & & \\
\hline
\end{tabular}
\caption{The errors, errors ratio of \( \eta_{2ih}(P) \) at points \( P = P_1, P_2 \).}
\end{table}
From Table 1-2, we can numerically see:
\[ \log_2 r^h (P) \approx 3 \text{, and } \log_2 \tau^h (P) \approx 5, \]
which shows that the convergent orders are three order for the approximation solutions \( \eta_{ih} \), and will be improved to five orders after EAs.

Acknowledgement

Project supported by the educational council foundation of Chongqing (KJ1500517), and natural science foundation of Chongqing (CSTC2013JCYJA00017)

References


