Pseudo Multi-hop Distributed Consensus Algorithm with Time Delay

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Abstract. In this paper, we analyze the pseudo multi-hop distributed consensus algorithm with time-delay. If there is time-delay among agents, the convergence performance of distributed consensus algorithms degrades. Supposing that the time-delay in every link is identical and invariant in time and spatial, we analyze the convergence performance of the pseudo multi-hop distributed consensus algorithm with time-delay, and calculate the maximum time-delay that the pseudo multi-hop algorithm remains stable. Finally, simulation results are provided to verify these analytical results.

Introduction

In the past few years, distributed consensus problems have attracted much attention from many fields. The distributed consensus algorithm is applied in many fields. Distributed average consensus means that each node can reach an agreement based on the information of local nodes. In [1, 2], a continuous first-order distributed consensus algorithm is proposed, and the theoretical framework for the analysis of consensus based on first order algorithms. Recently, many distributed consensus algorithms are proposed [3-7].

In order to accelerate the convergence rate of distributed consensus, in [7], the pseudo multi-hop distributed consensus algorithm under undirected graph is proposed. In [8], the pseudo multi-hop distributed consensus algorithm under directed topology is analyzed. The pseudo multi-hop distributed consensus algorithm accelerates dramatically the convergence rate by utilizing the previous information of non-adjacency nodes based on single-hop communication. The convergence rate of the pseudo multi-hop algorithm is much bigger than those of the other algorithms.


In the paper, we analyze the pseudo multi-hop algorithm with communication time-delay under directed topology and undirected topology. We analyze the stability of the pseudo multi-hop distributed consensus algorithm with communication time-delay, and calculate the maximum time-delay ensuring the pseudo multi-hop algorithm stable. For simplicity’s sake, we suppose that the communication time-delay in every links is identical and invariant in time and spatial.

Pseudo Multi-hop Distributed Consensus Algorithm with Time-delay

We introduce some notations and concepts that will be used through this paper. A graph $G_t = (V_t, E_t)$ represents the single-hop communication topology in a networked system with $N$ nodes, where $V_t$ is a set of vertices, and $E_t$ is a set of edges. Each edge in the graph is denoted by $(i, j)$,
where \( i,j \in V \). A directed graph is denoted by \((i,j) \neq (j,i)\). An adjacency matrix for directed graph \( G \) is denoted as \( A = \{a_{ij}\}_{N \times N} \), where \( a_{ij} = \begin{cases} 0 & (i,j) \in E_i \\ \neq 0 & (i,j) \notin E_i \end{cases} \). A set \( N_i \) is denoted as the set of vertices that can send directly information to vertex \( i \). In-degree of node \( i \) is \( \sum_{j \in N_i} a_{ji} \), and out-degree of node \( i \) is \( \sum_{j \in N_i} a_{ij} \). A path is a set of vertices, in which the edge formed by adjacent vertices is an element of the set \( E_1 \). A path is an m-hop path if it has \( m \) non-intersected edges. A graph \( G_m = (V_m, E_m) \) is defined as m-hop topology graph, where all elements of the edge set \( E_m \) are m-hop paths of the graph \( G_1 \), and \( V_m = V_1 \). An adjacency matrix of \( G_m \) is denoted as \( A_m = \{\beta_{ij}\}_{N \times N} \), where 
\[
\beta_{ij} = \begin{cases} a_{ij} & \text{if } (j,i) \text{ and } (i,s) \in E, \\ 0 & \text{otherwise} \end{cases} 
\]
A single-hop Laplacian matrix is denoted as \( L_1 = \{l_{ij}\} = \Delta_1 - A_1 \), where \( \Delta_1 = \text{diag}(\sum_{j=1}^{N} a_{ji}) \). An m-hop Laplacian matrix is denoted as \( L_m = \{l_{ij}\} = \Delta_m - A_m \), where \( \Delta_m = \text{diag}(\sum_{j=1}^{N} \beta_{ij}) \).

A directed graph is strong connected while there is a directed path between any two nodes, and a directed graph is balanced while the in-degree of every node is equal to itself out-degree, i.e. \( \deg_i = \deg_{out}(i), \forall i \in V \).

The pseudo multi-hop distributed consensus algorithm under directed graph or undirected graph in [7, 8] can be written as:

\[
\begin{align*}
    x_i(k+1) &= x_i(k) - \varepsilon \sum_{j=1}^{N} a_{ji}(k)\{x_j(k) - x_j(k)\} + y_j(k) \\
    &+ \sum_{j \in N_i} a_{ji}(k)\{x_j(k-1) - x_j(k-1)\} + y_j(k-1) \\
    &+ \sum_{j \in N_i} a_{ji}(k)\{x_j(k-2) - x_j(k-2)\} + \cdots \\
    &+ y_j(k-m+2) + \sum_{j \in N_i} a_{ji}(k)\{x_j(k-m+1) \\
    &- x_j(k-m+1)\} \\
    &= \sum_{j \in N_i} a_{ji}(k)\{x_j(k) - x_j(k)\} \\
    \end{align*}
\]

(1)

\[
\begin{align*}
y_i(k+1) &= \sum_{j \in N_i} a_{ji}(k)\{x_j(k) - x_j(k)\}
\end{align*}
\]

(2)

where \( x_i(k) \) is the state of node \( i \) in step \( k \), \( y_i(k) \) is the state derivative of node \( i \) in step \( k \), \( \varepsilon \) is a constant step size.

For simplicity’s sake, we suppose that the communication time-delay in every links is identical and invariant in time and spatial, and the communication time-delay is \( \tau \).

The pseudo multi-hop algorithm with communication time-delay can be written by:

\[
\begin{align*}
    x_i(k+1) &= x_i(k) - \varepsilon \sum_{j \in N_i} a_{ji}(k)\{x_j(k-\tau) - x_j(k-\tau)\} \\
    &+ y_j(k-\tau) + \sum_{j \in N_i} a_{ji}(k)\{x_j(k-1-2\tau) - x_j(k-1-2\tau)\} \\
    &+ \cdots + y_j(k-m+2-(m-1)\tau) + \sum_{j \in N_i} a_{ji}(k-m+1-\tau) \\
    &- x_j(k-m+1-\tau) \\
    &= \sum_{j \in N_i} a_{ji}(k)\{x_j(k) - x_j(k)\} \\
\end{align*}
\]

(3)
Dynamic Analyses for Pseudo Multi-hop Consensus Algorithm with Communication Time-delay

From (3) and (4), the collective dynamics of the group of nodes can be written as:

\[ x(k + 1) = x(k) - \sum_{\ell = 1}^{\infty} e^{-\ell \tau} x(k - \ell \tau) - e^{-\tau} \sum_{k_{\ell} \in \mathcal{N}_i} a_{ij} y_{ij}(k) \]

(4)

Form (5), we can get:

\[ x(k + 1) = x(k) - \sum_{\ell = 1}^{\infty} e^{-\ell \tau} x(k - \ell \tau) - e^{-\tau} \sum_{k_{\ell} \in \mathcal{N}_i} a_{ij} y_{ij}(k) \]

(5)

We can infer from (6):

\[ (z - 1 + e^{-\tau} L_1 - e^{-\tau} L_2 - e^{-\tau} L_3 - \cdots - e^{-\tau} L_m) x(k) = 0 \] (6)

The characteristic equation of system in (7) is given by:

\[ \det(I + \frac{e^{-\tau} L_1}{z - 1} - e^{-\tau} L_2 z^{-m \tau} - e^{-\tau} L_3 z^{-2m \tau} - \cdots - e^{-\tau} L_m z^{-m \tau}) = 0 \]

(7)

Obviously, if the roots of (8) have modulus less than unity except for a root at \( z = 1 \), then system in (3) and (4) with a connected undirected or directed balanced graph containing a globally reachable node achieves a consensus asymptotically.

But the calculation about the roots in (8) is very difficult. For simplicity, we calculate approximately the maximum time-delay in (5) by a simply way.

If the pseudo multi-hop algorithm with time-delay can reach a consensus, i.e.

\[ \lim_{k \to \infty} x(k + 1) = \lim_{k \to \infty} x(k) = \frac{1}{n} \mathbf{1}^T x(0) \]

(9)

where \( \mathbf{1} \) is a vector, \( \mathbf{1} = [1, 1, \cdots, 1]^T \). Obviously, we can get

\[ \lim_{k \to \infty} x(k - m \tau) = \lim_{k \to \infty} x(k - m + m \tau) \]

We can get from (5):

\[ \lim_{k \to \infty} x(k + 1) \approx \lim_{k \to \infty} x(k) - \sum_{\ell = 1}^{\infty} e^{-\ell \tau} x(k - \ell \tau) - e^{-\tau} \sum_{k_{\ell} \in \mathcal{N}_i} a_{ij} y_{ij}(k) \]

(10)

Obviously, equation (10) is the expression of the multi-hop algorithm, so the maximum time-delay in the pseudo multi-hop algorithm is equal to the maximum time-delay in the multi-hop algorithm.

We have the following important result.

**Theorem1.** For the pseudo multi-hop distributed consensus algorithm with communication time-delay in (5), when communication time-delay \( \tau \leq \tau^* \), the pseudo multi-hop distributed consensus algorithm with time-delay can reach an average consensus, where \( \tau^* \) is the maximum time-delay in multi-hop algorithm.

**Proof.** Based on the above mentioned analysis, we can get the results.

**Simulation Results**

In order to verify the results of the pseudo four-hop distributed consensus algorithm with time-delay, we test it on two different networks listed in figure 1, figure 2, denoted as \( G_1 \) and \( G_2 \). Topology \( G_1 \) is an undirected regular graph, and \( G_2 \) is a directed random network with 25 nodes. For the sake of simplicity, we assume the weights \( a_{ij} = 1 \) for any links, and the initial states of nodes is stochastic variable range from 0 to 50.
Figure 1. Undirected regular topology graph.

Figure 2. Undirected random topology graph.

Figure 3. The convergence performance about the pseudo four-hop distributed consensus algorithm with time-delay under graph $G_1$. 
Figure 4. The convergence performance about the pseudo four-hop distributed consensus algorithm with time-delay under graph $G_2$.

Figure 5 Comparison of the maximum time-delays in the pseudo four-hop algorithm and in four-hop algorithm under graph $G_1$.

Figure 6. Comparison of the maximum time-delays in the pseudo four-hop algorithm and in four-hop algorithm under graph $G_5$.

From figure 3 to figure 6 show the simulation results of the mean-square error of the pseudo four-hop distributed consensus algorithm under $G_1$ and $G_2$ with the random initial state from 0 to 50, and step size $\varepsilon = 0.02$. Obviously, the convergence performance of the pseudo four-hop distributed consensus algorithm degrades when variance of communication noise increases, and the mean-square error of pseudo four-hop distributed consensus algorithm increases when variance of communication noise increases.
Conclusions

In this paper, we analyze the convergence performance of the pseudo multi-hop distributed consensus algorithm with communication time-delay. When there is communication time-delay among agents, the performance of distributed consensus algorithm degrades. We analyze the convergence performance of the pseudo multi-hop distributed consensus algorithm with time-delay, and calculate the maximum time-delay. By calculating, the maximum time-delay in the pseudo multi-hop algorithm is equal to the maximum time-delay in the multi-hop algorithm.

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