The Optimization of the Mean-Variance Portfolio Selection with Nonsmooth Concave Transaction Costs

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Abstract. Transaction cost was an important factor in the financial market. According to the situation of the underdeveloped market, we proposed a mean-variance portfolio selection model with nonsmooth concave transaction costs. This is nonsmooth programming problem. The paper proposed the subsection method and pivoting algorithm to solve it, and proved these algorithms convergent. Finally, according to annual yields of six kinds of securities in eight years, we solved the mean-variance portfolio optimal strategy with nonsmooth concave transaction costs. The example was given to illustrate the efficiency of the algorithms, which offered a new way to solve the problems of nonsmooth programming.

Introduction

Markowitz [1,2] proposed the standard mean-variance model which laid the foundation of the optimal portfolio strategy. The model introduced that the variance measured risks. However, the model didn’t consider the transaction cost in the process of investment. In fact, transaction cost is one of the important factors in the financial market, and neglecting it could lead to the inefficient portfolio selection (Yoshimoto [3]). In recent years, many scholars discussed the portfolio selection of transaction cost. For example, Jason [4] studied three kinds of mean-variance portfolio models with no-convex transaction cost by heuristic. Li and Wang [5], Tang [6], Rong et al. [7] studied the optimized solution of portfolio selection model with linear transaction cost by using interactive method, tree algorithm and fuzzy algorithm respectively. Zhang and Zhang [8] studied the mean-variance portfolio selection model of convex transaction cost with the application of pivoting algorithm.

In the actual market, the trading volume is in inverse proportion to the unit transaction cost. And unit transaction cost declines with the rise of volume. In other words, transaction cost function is a nonsmooth (continuous and non-differentiable) concave function. This paper proposed the mean-variance portfolio model with the nonsmooth concave transaction costs. Meanwhile this is nonsmooth programming problem. We will solve the problem by using subsection method and pivoting algorithm.

Symbols and Introductions

As a matter of convenience, we assume that assets are infinitely divisible. With n assets to choose from, the yield of the i-th asset is $R_i$ (random variable). And its expectation is $r_i = E(R_i)$, setting $R = (R_1, R_2, ..., R_n)^T$, $r = (r_1, r_2, ..., r_n)^T$. Covariance matrix is $G = (\sigma_{ij})_{n \times n}$, $\sigma_{ij} = COV(R_i, R_j)$, $i, j = 1, 2, ..., n$. The investment proportion of the i-th asset is $x_i$, $i = 1, 2, ..., n$. We denote capital budget constraint by $x = (x_1, x_2, ..., x_n)^T$, $x_1 + x_2 + ... + x_n = 1$. The yield of portfolio selection is $R_p = R^T x$. Its mean is $r_p = r^T x$ and variance is $\sigma_p^2 = x^T G x$, especially $r_p \geq \min \{r_1, r_2, ..., r_n\}$. $c_i(x_i)$ is the transaction cost of the i-th asset.

**Definition 1.** Let $c(x)$ be a function of a convex set $D$ on $\mathbb{R}^n$. If any two points $x^{(1)}, x^{(2)}$ and real number $a, (0 < a < 1)$ in $D$, there is
We will say that \( c(x) \) is the concave function defined on \( D \).

**The Mean-variance Portfolio Model with the Nonsmooth Concave Transaction Cost**

Assuming transaction cost function of the \( i \)-th asset is nonsmooth concave, as shown in Figure 1.

![Figure 1. Nonsmooth concave transaction costs function.](image1)

A mean-variance portfolio model with nonsmooth concave transaction costs is shown as follows:

\[
\begin{align*}
\min f(x) &= \lambda^{T} G x - (1 - \lambda^{T}) \sum_{i=1}^{n} [r_{i} x_{i} - c_{i}(x_{i})] \\
\text{s.t.} & \sum_{i=1}^{n} x_{i} = 1 \\
& 0 \leq x_{i} \leq a_{i}, \quad i = 1, 2, \ldots, n
\end{align*}
\]

Model (1) is a nonsmooth concave programming problem with linear constraints. \( c_{i}(x_{i}) \) which can be fit by linear function, is a non-differential and concave function, as shown in Figure 2.

![Figure 2. The transaction cost function.](image2)

where line \( OB \) is \( k_{i}(x_{i}) = k_{i} x_{i} \), \( k_{i} = c_{i}(a_{i})/a_{i} \)

If we substitute \( k_{i}(x_{i}) \) for \( c_{i}(x_{i}) \), model (1) could transform into the model which is shown as follows:

\[
\begin{align*}
\min g_{o}(x) &= \lambda^{T} G x - (1 - \lambda) \sum_{i=1}^{n} [r_{i} x_{i} - k_{i} x_{i}] \\
\text{s.t.} & \sum_{i=1}^{n} x_{i} = 1 \\
& 0 \leq x_{i} \leq a_{i}, \quad i = 1, 2, \ldots, n
\end{align*}
\] (2)
**Theorem 1.** Let \( x \) respectively be the feasible solution of model (1) and model (2), then

\[
g_0(x) \leq f(x) \tag{3}
\]

Proof. Because \( c_i(x_i) \) is a concave function, \( c_i(x_i) \geq k_i x_i \).

Therefore

\[
\sum_{i=1}^{n} c_i(x_i) \geq \sum_{i=1}^{n} k_i x_i,
\]

i.e., \( g_0(x) \leq f(x) \).

Theorem has been proved.

Assuming the optimal solution of model (2) is \( x^0=(x_1^0, \ldots, x_n^0)^T \). If

\[
\sum_{i=1}^{n} [c_i(x_i^0) - k_i x_i^0] \leq \varepsilon
\]

then \( x^0 \) is the approximate solution of model (1). What’s more, the error is less than or equal to \( \varepsilon \).

**Theorem 2.** Let \( x^*, x^0 \) respectively be the optimal solution of model (1) and model (2), then

\[
g_0(x^*) \leq f(x^*) \leq f(x^0) \tag{5}
\]

Proof. Because \( x^0 \) is the optimal solution of model (2), \( x^0 \) is the feasible solution of model (1). And \( x^* \) is the optimal solution of model (1), so \( f(x^*) \leq f(x^0) \). \( x^* \) is the feasible solution of model (2), so we could know \( g_0(x^*) \leq f(x^*) \) according to Theorem 1. As a result, \( g_0(x^*) \leq f(x^*) \leq f(x^0) \), i.e., the solution of model (1) is convergent.

Theorem has been proved.

If formula (3) is false, we let

\[
c_s(x_s) - k_s x_s^0 = \max\{c_i(x_i^0) - k_i x_i^0\}
\]

and let

\[
X_1=\{x / 0 \leq x_s \leq as/2, \ 0 \leq x_j \leq a_j, \ j \neq s\}
\]

\[
X_2=\{x / as/2 \leq x_s \leq as, \ 0 \leq x_j \leq a_j, \ j \neq s\}
\]

The transaction cost function of \( x_s \) can use linear function of \( c_s(x_s) \) and \( c_s(x_s) \) to fit, as shown in Figure 3.

![Figure 3 The transaction cost function](image)

where line OA is \( c_{s1}(x_s) \), whose slope is \( k_{s1} = c_{s1}(x_s/2)/(x_s/2) \). And line AB is \( c_{s2}(x_s) \), whose slope is \( k_{s2} = (c_s(a_s) - c_s(a_s/2))/(a_s/2) \). Then model (2) could transform into two models which are shown as follows:

\[
\min g_1(x) = \lambda x^T G x - (1-\lambda) \left\{ \sum_{i=1}^{n} [r_i x_i - k_i x_i] + r_s x_s - c_{s1}(x_s) \right\}
\]
Theorem 3. Let $x^1$ be the optimal solution of model (9), then

$$g_0(x^*) \leq g_1(x^1)$$

Proof. Because $x^1$ is the optimal solution of model (9), $x^1$ is the feasible solution of model (9). And $k_i x^1 \leq c_i(x^1)$, so

$$g_0(x^1) \leq g_1(x^1)$$

Because $x^*$ is the optimal solution of model (2), $x^1$ is the feasible solution of model (2). Thus

$$g_0(x^*) \leq g_0(x^1)$$

According to formula (12) and formula (13), we could know $g_0(x^*) \leq g_1(x^1)$

Theorem has been proved.

According to Theorem 3, the objective function will get larger if we substitute $c_i(x^1)$ and $c_i(x^2)$ for $c_i(x^*)$. $x^1$ is the approximate optimal solution of model (1) on condition that $\sum_{i=1}^{n}[c_i(x^1) - k_i x^1] \leq \varepsilon$ ($\varepsilon$ is any small positive number).

Algorithms 1. Computation steps of model (1) is shown as follows:

Step 1: $P \{ (P_0) \}, \quad \hat{f} = +\infty, \quad k = 0, \quad b^0 = 0, \quad a^0 = a, \quad X_0 = \{ x \mid b_i \leq x \leq a_i \}.$

Step 2: If $P = \{ \emptyset \}$, then turn Step 9, otherwise turn Step 3.

Step 3: Choose a problem $(P_k) \in P$

$$\min f(x)$$

$$(P_k) \quad \begin{cases} x_1 + \cdots + x_n = 1 \\ x \in X_k \end{cases}$$

$P = P \setminus \{(P_k)\}$

Step 4: Approximately estimate $c_i(x_j)$ through $k_i(x_j)$, and $X_k = \{ x_i \mid b_i \leq x_i \leq a_i \}$. The following mathematical programming can be transformed into:

$$\min g_k(x) = \lambda x^T G x - (1 - \lambda) \sum_{i=1}^{k} (r_i x_i - k_i(x_i))$$

$$(Q_k) \quad \begin{cases} x_1 + \cdots + x_n = 1 \\ x \in X_k \end{cases}$$
Solve $Q_k$ by using pivoting algorithm. Turn Step 2 if $Q_k$ doesn’t have feasible solution, otherwise we should let $x^k$ be the optimal solution of $Q_k$. Turn Step 8 if $\sum_{i=1}^n [c_i(x_i) - k_i(x_i)] > \varepsilon$, otherwise $f_k = f(x^k)$

**Step 5**: Turn Step 7 if $f_k > \hat{f}$, otherwise Step 6.

**Step 6**: $\hat{f} = f_k$ and $\hat{x} = x^k$, what’s more, delete all sub problems about $g_k(x) \geq \hat{f}$.

**Step 7**: Turn Step 2 if $g_k(x^k) \geq \hat{f}$, otherwise turn Step 8.

**Step 8**: Make the following assumptions: $c_i(x_i) - k_i x_i = \max \{c_i(x_i) - k_i x_i : i=1, \ldots, n\}$, $X_{i+1} = X_i \cap \{b^k \leq x_i \leq (a_i^k + b_i^k) / 2\}$, $X_{i+2} = X_i \cap \{(a_i^k + b_i^k) / 2 \leq x_i \leq a_i^k\}$, and define two subproblems:

$$
\begin{align*}
(P_{i+1}) & \quad \min f(x) \\
& \quad \text{s.t.} \begin{cases} 
  x_1 + \cdots + x_n = 1 \\
  x \in X_{i+1} 
\end{cases} \\
(P_{i+2}) & \quad \min f(x) \\
& \quad \text{s.t.} \begin{cases} 
  x_1 + \cdots + x_n = 1 \\
  x \in X_{i+2} 
\end{cases}
\end{align*}
$$

$P = P \cup \{P_{i+1}, P_{i+2}\}$, $k = k+1$, and turn Step 3.

**Step 9**: Stop calculating and $\hat{x}$ is the optimal solution of model (3).

**Example**

Annual yields of six kinds of securities in eight years are showed as Table 1. Solve the mean-variance portfolio optimal strategy with nonsmooth concave transaction costs, where the transaction cost function is

$$c_i(x_i) = \begin{cases} 
  0.01 x_i, & 0 \leq x_i \leq 0.5 \\
  0.015 x_i + 0.00025, & 0.5 \leq x_i \leq 0.8 \\
  0.02 x_i + 0.0004, & 0.8 \leq x_i \leq 1 
\end{cases}$$

<table>
<thead>
<tr>
<th>Bargain Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Security 1</td>
<td>0.04</td>
<td>0.07</td>
<td>0.09</td>
<td>0.13</td>
<td>0.14</td>
<td>0.17</td>
<td>0.21</td>
<td>0.24</td>
</tr>
<tr>
<td>Security 2</td>
<td>0.14</td>
<td>0.06</td>
<td>0.08</td>
<td>0.15</td>
<td>0.11</td>
<td>0.13</td>
<td>0.10</td>
<td>0.11</td>
</tr>
<tr>
<td>Security 3</td>
<td>0.13</td>
<td>0.13</td>
<td>0.11</td>
<td>0.15</td>
<td>0.10</td>
<td>0.07</td>
<td>0.14</td>
<td>0.11</td>
</tr>
<tr>
<td>Security 4</td>
<td>0.12</td>
<td>0.04</td>
<td>0.18</td>
<td>0.13</td>
<td>0.19</td>
<td>0.16</td>
<td>0.14</td>
<td>0.11</td>
</tr>
<tr>
<td>Security 5</td>
<td>0.18</td>
<td>0.06</td>
<td>0.22</td>
<td>0.15</td>
<td>0.14</td>
<td>0.06</td>
<td>0.08</td>
<td>0.09</td>
</tr>
<tr>
<td>Security 6</td>
<td>0.15</td>
<td>0.04</td>
<td>0.08</td>
<td>0.06</td>
<td>0.13</td>
<td>0.05</td>
<td>0.10</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Table 1. Annual yields of six kinds of security in Eight years.

Solve: Calculate the arithmetic average of yield on each security, as shown in Figure 2:

<table>
<thead>
<tr>
<th>Security</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>The average annual return</td>
<td>0.13625</td>
<td>0.11</td>
<td>0.1175</td>
<td>0.13375</td>
<td>0.1225</td>
<td>0.0875</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. The arithmetic average of yields on securities.

Calculate covariance matrix of samples, the results are as follows:
G = 
\[
\begin{bmatrix}
0.0047 & 0.0002 & -0.0004 & 0.0006 & -0.0021 & -0.0003 \\
0.0002 & 0.0009 & 0.0000 & 0.0005 & 0.0003 & 0.0004 \\
-0.0004 & 0.0000 & 0.0007 & -0.0005 & 0.0003 & 0.0001 \\
0.0006 & 0.0005 & -0.0005 & 0.0022 & 0.0013 & 0.0007 \\
-0.0021 & 0.0003 & 0.0003 & 0.0013 & 0.0035 & 0.0011 \\
-0.0003 & 0.0004 & 0.0001 & 0.0007 & 0.0011 & 0.0015
\end{bmatrix}
\]

Calculate the optimal investment strategy of $r_0$ in different values respectively, as shown in Table 3:

<table>
<thead>
<tr>
<th>Optimal Solution</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$\sigma_p^2$</th>
<th>$\lambda_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0762</td>
<td>0.1711</td>
<td>0.5475</td>
<td>0.1742</td>
<td>0.0000</td>
<td>0.0310</td>
<td>0.1202</td>
<td>0.0003</td>
<td></td>
</tr>
<tr>
<td>0.3729</td>
<td>0.0000</td>
<td>0.5254</td>
<td>0.0000</td>
<td>0.1017</td>
<td>0.0000</td>
<td>0.1250</td>
<td>0.0006</td>
<td></td>
</tr>
<tr>
<td>0.1609</td>
<td>0.0000</td>
<td>0.3649</td>
<td>0.4742</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1300</td>
<td>0.0006</td>
<td></td>
</tr>
<tr>
<td>0.3729</td>
<td>0.0000</td>
<td>0.0254</td>
<td>0.5000</td>
<td>0.1017</td>
<td>0.0000</td>
<td>0.1350</td>
<td>0.0014</td>
<td></td>
</tr>
<tr>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1363</td>
<td>0.0047</td>
<td></td>
</tr>
</tbody>
</table>

From Table 3, the optimal portfolio selection is 37.29%, 52.54% and 10.17% respectively to the first asset, the third asset, the fifth asset. Meanwhile, we could figure out the optimal investment strategy corresponding to expected yield of the other portfolio selection.

**Conclusions**

Transaction costs are widespread, and its function is non-smooth concave normally in the situation of the underdeveloped market, thus the research makes investment analysis closer to reality. The paper proposed a mean-variance portfolio selection model with non-smooth concave transaction costs. This is non-smooth programming problem. Combining subsection method with pivoting algorithm, we solved it and proved these algorithms is convergent, which offers a new way to solve the problems of non-smooth programming.

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**References**


