Integration of Heterogeneous Complex Variable Function and Combination Formula

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Abstract. In this paper, we discuss the integration of heterogeneous complex variable function \( f(c) = \frac{1}{(c-a)^2} \) in curve \( C \) of heterogeneous complex plane \( H_K \) by using heterogeneous Cauchy integral theorem, Cauchy integral formula and residue theory. And then obtain two identical equation of combinatorial mathematics by using two different calculation methods.

Introduction

In the references [1, 2], many authors introduce some reaction diffusion models of biology, study the diffusion problem of heterogeneous manifolds with homogeneous bottom space. Considering these heterogeneous problems, they always discuss the heterogeneous plexus space in the homogeneous bottom space, which brings a lot of difficulties to the research for a long time. In order to solve the above problems, the references [3,4,5] proposed a heterogeneous complex variable function theory, and gave the definition of heterogeneous complex domain, heterogeneous analytic function and heterogeneous integral. Based on this, the relationship between the heterogeneous Laplace equation and the heterogeneous Cauchy-Riemann equations is established. Accordingly, we can obtain the heterogeneous Cauchy integral theorem and the Cauchy integral formula. It’s so important to calculate the expansion coefficient when facing the Taylor expansion and Laurent series expansion of the heterogeneous analytic function. In this process, the key problem is the calculation of a common integral \( \int_{C} \frac{de}{(z-a)^2} \), where the \( C \) represents the path of integration in the heterogeneous complex plane \( H_K \); a circle taking as \( a \) center and \( \rho \) as radius. Therefore, with the help of the heterogeneous Cauchy integral theorem, the Cauchy integral formula and Residue theory in the references [4,5,6,7], we take two different calculation methods to discuss the above integral and thus obtain two identical equation in combinatorial mathematics.

Preliminaries

Some basic definitions and properties of heterogeneous complex number and heterogeneous complex variable function are given below, which can also be seen in the references [3,4, 5] specifically.

Definition 2.1. The heterogeneous complex numbers field \( C_k \) consists, by definition \( H_k = \{ z | z = a+jb \} \) where \( a, b \in R, j^2 = -k, k > 0 \), and the vector operations are defined by
\[ z = a + jb, m \in R, \text{ then, } ma = ma + jmb, z_1 = a_1 + jb_1 \text{ and } z_2 = a_2 + jb_2, \text{ then } \]
\[ z_1 + z_2 = a_1 + a_2 + j(b_1 + b_2). \]

If we give a product operation by
\[ z_1 = a_1 + jb_1 \text{ and } z_2 = a_2 + jb_2, \text{ then } z_1z_2 = a_1a_2 - kb_1b_2 + j(a_1b_2 + a_2b_1). \]

And the length of \( z = a + jb \) be defined by \( |z| = \sqrt{a^2 + kb^2}, \) and then inverse element of \( z = a + jb \) be \( z^{-1} = \frac{a - jb}{|z|^2}, \) hence \( H_k \) is a field. It is called heterogeneous complex numbers field.

**Definition 2.2.** Let \( f \) be function from \( H_k \) to \( H_k, \) \( z_0 \) be a fix point in \( C_k, \) \( f \) be called differentiable function in \( z_0 \), if the following limit exists:
\[ A = \lim_{\Delta z \to 0} \frac{f(z_0) - f(z)}{\Delta z}, \]
and then \( A \) be called derivative of \( f(z) \) in \( z_0 \) by being denoted by \( f'(z_0). \)

Suppose \( f \) be a function in set \( D, \) \( f \) be called a differentiable function in \( D, \) if every \( z \in D, \)
\( f(z) \) is differentiable.

**Definition 2.3.** Suppose a directed curve \( C_j \) in \( H_k \) starting from \( a = z(\alpha) \) and ending in \( b = z(\beta), \)

of the form
\[ z = z(t) \quad (\alpha \leq t \leq \beta) \]
\( f(z) \) is defined along \( C_j, \) we take some points of division between \( a \) and \( b: \)
\[ a = z_0, z_1, \ldots, z_n = b, \]
then \( C_j \) can be divided into \( n \) segmental arcs, for \( \forall \xi_i \in [z_{i-1}, z_i], \) we have
\[ S_n = \sum_{i=1}^{n} f(\xi_i) \Delta z_i, \]
\[ \Delta z_i = z_i - z_{i-1}. \]

When the number of points increases, the arc segments are gradually fined. If the limit of sum number \( S_n \) exists and denote \( S, \)
then \( f(z) \) is said to be integrable along \( C_j, \) the value is \( S, \) \( C_j \) is integral path. That is
\[ S = \int_{C_j} f(z)dz. \]

**Theorem 2.4.** Suppose function \( f(z) \) is analytical in a simply connected region \( D, \)
\( C_j \) is a random contour or closed curve in \( D, \) then
\[ \int_{C_j} f(z)dz = 0. \]
Theorem 2.5. Suppose the boundary of domain D in $\mathbb{H}_k$ be closed curve $C_j$, $f(z)$ be an analytic function in domain D and continuous function on $\tilde{D} = D + C_j$, then

$$f(z) = \frac{1}{2\pi i} \oint_{C_j} \frac{f(\xi)}{\xi - z} \, d\xi, \quad (z \in D).$$

Theorem 2.6. Suppose the boundary of domain D in $\mathbb{H}_k$, $C_j$ be closed curve in $\mathbb{H}_k$, $f(z)$ be an analytic function in domain D and continuous function on $\tilde{D} = D + C_j$, then there exists each order derivative for $f(z)$ and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{C_j} \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi, \quad (z \in D).$$

Two Integral Methods and Combination Formula

In this section, we will use the theory of complex variable function and the theory of heterogeneous complex variable function in section 2 to deal with important integral

$$\int_{(z-a)} \frac{dz_k}{(z_k - a)^n},$$

where $C_j$ denote the circle $|z-a|=\rho$ centered at the point $a = x_0 + jy_0$ and with radius $\rho$ in the heterogeneous complex plane. And then obtain two identical equation of combinatorial mathematics by using two different calculation methods if $n \geq 3$.

According to theorem 2.5, theorem 2.6, we have,

$$\int_{C_j} \frac{dz_k}{(z_k - a)^n} = \begin{cases} 2\pi i, n = 1, \\ 0, \ n \neq 1. \end{cases} \quad (3.1)$$

when $n = 1,2$, by using residue theory, the above equations (3.1) holds. We use residue theory to obtain two combination formula by calculate the left side of (3.1) if $n \geq 3$,

$$C_j = \{z_k \ | \ |z_k - a| = \rho, z_k \in \mathbb{H}_k\}.$$ 

$$z_k - a = \rho(\cos \theta + j \sin \theta), 0 \leq \theta \leq 2\pi, \quad \rho = \sqrt{(x-x_0)^2 + (y-y_0)^2}, \quad \cos \theta = \frac{x-x_0}{\rho}, \quad \sin \theta = \frac{y-y_0}{\rho},$$

where

$$z = x + jy, a = x_0 + jy_0.$$ Firstly, let $F(z_k)$ denote the function $\frac{1}{(z_k - a)^n}$.

Let $z = e^{i\theta}$, then

$$\cos \theta = \frac{z + z^{-1}}{2}, \quad \sin \theta = \frac{z - z^{-1}}{2i}, \quad d\theta = \frac{dz}{iz}.$$ 

If $n > 2$, we calculate
\[ \int_{C_j} \frac{dz}{(z_k-a)^n} = \int_{|z| = 1} \frac{2^{n-1}[(\sqrt{k} + 1) z^2 + (\sqrt{k} - 1)] z^{n-2}}{\rho^{n-1}[(\sqrt{k} + 1) z^2 - (\sqrt{k} - 1)]} \, dz = \int_{|z| = 1} h(z) \, dz \]

Where \( h(z) = \frac{2^{n-1} \left( \frac{z^2 + \sqrt{k} - 1}{\sqrt{k} + 1} \right) z^{n-2}}{(\sqrt{k} + 1)^{n-1} \rho^{n-1} \left( z^2 - \frac{\sqrt{k} - 1}{\sqrt{k} + 1} \right)^n} \), we denote \( A \) by \( \frac{\sqrt{k} - 1}{\sqrt{k} + 1} \) and \( B \) by \( \frac{2^{n-1}}{(\sqrt{k} + 1)^{n-1} \rho^{n-1}} \).

Then

\[ h(z) = \frac{2^{n-1} (z^2 + A) z^{n-2}}{(\sqrt{k} + 1)^{n-1} \rho^{n-1} (z^2 - A)^n} = \frac{Bz^n}{(z^2 - A)^n} + \frac{ABz^{n-2}}{(z^2 - A)^n} = h_1(z) + h_2(z), \]

(a) When \( k > 1 \), that is \( 0 < A < 1 \), then in the interior of a circle \( |z| = 1 \),

\[ h_1(z) = \frac{Bz^n}{(z + \sqrt{A})^n (z - \sqrt{A})^n} \]

have 2 poles of order one: \( z_i = -\sqrt{A} \) and \( z_2 = \sqrt{A} \); have 2 poles of order one:

\[ z_3 = -\sqrt{A} \text{ and } z_4 = \sqrt{A}. \]

Calculating their residues

Therefore,

\[ \int_{|z| = 1} h(z) \, dz = \frac{2\pi i}{\sqrt{k}} \left( \text{Res}_{z_i} h_1(z) + \text{Res}_{z_2} h_1(z) + \text{Res}_{z_3} h_2(z) + \text{Res}_{z_4} h_2(z) \right) \]

\[ = \frac{2\pi i}{\sqrt{k}} \sum_{m=0}^{n-2} \frac{C_{n-1}^m C_{2n-m-3}^{n-1} 2^{-2n+m+1}}{(n-m)(n-m-1)} \begin{vmatrix} n(2n-2m-2) \left[ (-1)^{m+1} + (-1)^{n-m-1} \right] \\ + 2(n-m) \left[ (-1)^{m+2} + (-1)^{n-m-2} \right] \end{vmatrix} \]

\[ + \frac{2\pi i}{\sqrt{k}} \sum_{m=0}^{n-1} \frac{C_{n-1}^m C_{2n-m+1}^{n+1} 2^{-2n+m+1}}{(n+1)^2} \begin{vmatrix} (n+2)(2n-m+2) \left[ (-1)^{m+1} + (-1)^{n-m+1} \right] \\ + 2(n-m+1) \left[ (-1)^{m+2} + (-1)^{n-m} \right] \end{vmatrix} \]

When \( 0 < k \leq 1 \), that is \( A \leq 0 \), Their residues ar siemilar (a).

Consequently, we can gain two combinatorial identical equations from the above.

**Theorem 3.1.** If \( n \geq 1 \), then

\[ \sum_{m=0}^{n} C_{n+1}^m C_{2n-m+1}^{n+1} 2^{-2n+m+3} \begin{vmatrix} (n+2)(2n-m+2) \left[ (-1)^{m+1} + (-1)^{n-m+1} \right] \\ + 2(n-m+1) \left[ (-1)^{m+2} + (-1)^{n-m} \right] \end{vmatrix} = (n+2)2^{-(n+2)} \left[ (-1)^n - 1 \right] \]

(3.2)
\[
\sum_{m=0}^{2n-1} C_{2n}^{2n} C_{4n-m-1}^{2n} 2^{-4n+m+1} \frac{(2n+1)(4n-m)}{(2n-m+1)(2n-m)} \left[ (-1)^{n+1} + (-1)^{2n-m} \right] \\
+ \frac{2(2n-m)}{(2n+2)^2} \left[ (-1)^{n+2} + (-1)^{2n-m-1} \right] + (2n+1)2^{-2n-1} = 0.
\]

(3.3)

**Summary**

In order to further study the Taylor expansion and Laurent expansion of the heterogeneous complex variable function, this paper uses the heterogeneous Cauchy integral formula and the heterogeneous integral theorem in the [3,4,5] of the analytical function, and the residue theory[7] of the complex variable function to calculate an important integral formula of the expansion, thus we establish two combinatorial identities. In addition, we have a lot of further discussion, such as how to establish the classification of singularities, similar residue theorem and the application of the heterogeneous complex number and variable function in physics, mechanics and e.t.c.

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**References**


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