The Method of Non-Local Control Improvement in Optimal Control Problems with Constraints

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Abstract. A new approach to solving optimal control problems with terminal and phase constraints is proposed on the basis of constructing and solving a system of conditions for improving admissible control in the form of the fixed point problem for a constructed control operator. This form makes it possible to apply the developed theory and fixed-point methods for the effective search of admissible improving controls. To construct these conditions, we apply the transition to the auxiliary problem of optimizing a regular Lagrange functional without constraints with mixed control functions and parameters. Approbation and comparative analysis of the effectiveness of the proposed approach of fixed points on model and test examples are carried out.

Introduction

A common approach to solving optimal control problems with constraints is a reduction to auxiliary problems without restrictions with the help of Lagrangian functionals, on the basis of which the necessary optimality conditions for the type of the maximum principle are obtained [1], [2], [10], [11]. In the classical form, the necessary optimality conditions are non-constructive, since the choice of Lagrange multipliers remains open.

The search for extremal controls satisfying the necessary optimality conditions is usually divided into the stages of searching for extremal control in an auxiliary problem without constraints with Lagrange multipliers and the stages of selecting these factors to satisfy the constraints. For an approximate solution of auxiliary Lagrange problems, methods of constructing relaxation sequences of controls, in particular, gradient methods and their modifications, are often used [10], [11].

In this paper, we propose a new method for improving controls, which allows us to build a relaxation sequence of admissible controls and to find an approximate solution of the problem with constraints. The method of improving control is based on constructing conditions for improving control with the exact fulfillment of constraints based on special increment formulas for the functional of the auxiliary Lagrange problem, which does not contain residual terms of expansions. The use of such formulas makes it possible to interpret the system of conditions for improving the admissible control as a fixed point problem. This allows us to apply the developed theory and fixed point methods for the effective search for admissible improvement controls.

The methods of fixed points were previously constructed and substantiated in classes of non-linear optimal control problems without constraints [4], [5], [6]. In this paper, these approaches are developed for problems with constraints.
Statement of the Problem with Constraints

We consider a class of problems with constraints that can be reduced to the following general form

\[
\dot{x}(t) = f(x(t), u(t), \omega, t), \quad x(t_0) = x^0, \quad u(t) \in U, \omega \in W, t \in T = [t_0, t_1],
\]

\[
\Phi_0(\sigma) = \varphi_0(x(t_1), \omega) + \int_T F_0(x(t), u(t), \omega, t)dt \to \inf_{\sigma \in \Omega},
\]

\[
\Phi_1(\sigma) = \varphi_1(x(t_1), \omega) = 0,
\]

in which \(x(t) = (x_1(t), \ldots, x_n(t))\) - the state vector, \(u(t) = (u_1(t), \ldots, u_m(t))\) - the vector of the control functions, \(\omega = (\omega_1, \ldots, \omega_l)\) - the vector of the control parameters. The sets \(U \subseteq \mathbb{R}^n, W \subseteq \mathbb{R}^l\) are closed and convex. The interval \(T\) is fixed. As the available control functions, we consider a set of \(V\) piecewise continuous functions on \(T\) with values in the set \(U\). \(\sigma = (u, \omega)\) - an available control with values in the set \(\Omega = V \times W\). The functions \(\varphi_0(x, \omega), \varphi_1(x, \omega)\) are continuously differentiable on \(\mathbb{R}^n \times W\), the functions \(F_0(x, u, \omega, t)\) and their partial derivatives with respect to \(x, u, \omega\) are continuous in the set of arguments on the set \(\mathbb{R}^n \times U \times W \times T\). The function \(f(x, u, \omega, t)\) satisfies the Lipschitz condition by \(x\) in \(\mathbb{R}^n \times U \times W \times T\) with the constant \(L > 0\): \(\|f(x, u, \omega, t) - f(y, u, \omega, t)\| \leq L\|x - y\|\).

The conditions guarantee the existence and uniqueness of the solution \(x(t, \sigma), t \in T\) of the system (1) for any available control \(\sigma \in \Omega\). The available control \(\sigma \in \Omega\) is called admissible if the functional constraint (2) is satisfied. We denote the set of admissible controls by \(D = \{\sigma \in \Omega : \Phi_1(\sigma) = \varphi_1(x(t_1), \omega) = 0\}\).

Optimal control problems with terminal, phase and mixed constraints, including non-fixed time, can be submitted to the form (1) - (3).

The problem of improving admissible control in the class of problems (1) - (3) is considered in the following general formulation: for a given admissible control \(\sigma' \in D\) is required to find an admissible control \(\sigma \in D\) with the condition \(\Delta_\sigma \Phi_0(\sigma') = \Phi_0(\sigma) - \Phi_0(\sigma') \leq 0\).

Method for Improving Control

We consider the auxiliary problem without constraints on the basis of the regular Lagrange functional

\[
\dot{x}(t) = f(x(t), u(t), \omega, t), \quad x(t_0) = x^0, \quad u(t) \in U, \omega \in W, t \in T = [t_0, t_1],
\]

\[
L(\lambda, \sigma) = \Phi_0(\sigma) + \lambda \Phi_1(\sigma) \to \inf_{\sigma \in \Omega}, \lambda \in \mathbb{R}.
\]

We denote \(\varphi(\lambda, x, \omega) = \varphi_0(x, \omega) + \lambda \varphi_1(x, \omega)\). The Pontryagin function with conjugate variable \(\psi \in \mathbb{R}^n\) and the standard conjugate system in problem (4), (5) have the form

\[
H(\psi, x, u, \omega, t) = \langle \psi, f(x, u, \omega, t) \rangle - F_0(x, u, \omega, t),
\]

\[
\dot{\psi}(t) = -H_x(\psi(t), x(t), u(t), \omega, t), \quad t \in T, \psi(t_1) = -\varphi_x(\lambda, x(t_1), \omega).
\]
For an available control \( \sigma \in \Omega \) we denote by \( \psi(t, \sigma, \lambda) \), \( t \in T \) - solution of the standard conjugate system (6) for \( x(t) = x(t, \sigma) \).

Consider the problem of improving the available control in task (4), (5): for a given available control \( \sigma^I \in \Omega \) it is required to find an available control \( \sigma \in \Omega \) with a condition \( \Delta_{\sigma} L(\lambda, \sigma^I) = L(\lambda, \sigma) - L(\lambda, \sigma^I) \leq 0 \). In accordance with [4], [6], the nonlocal conditions for improving the available control \( \sigma^I \in \Omega \) based on the use of special formulas for the increment of the functional without the remainder terms of the expansions can be represented as follows.

Next, we use the following notation for a particular increment an arbitrary vector-valued function \( g(y_1, \ldots, y_t) \) with respect to \( y_{s_1}, y_{s_2} \)

\[
\Delta_{y_{s_1}} + \Delta_{y_{s_1}} + \Delta_{y_{s_2}} g(y_1, \ldots, y_t) = \\
g(y_1, \ldots, y_{s_1} + \Delta_{y_{s_1}}, \ldots, y_{s_2} + \Delta_{y_{s_2}}, \ldots, y_t) - g(y_1, \ldots, y_t).
\]

In addition, we denote \( \Delta x(t) = x(t, u) - x(t, u^I) \), \( \Delta u(t) = u(t) - u^I(t) \).

We introduce a modified differential-algebraic conjugate system including an additional phase variable \( y(t) = (y_1(t), \ldots, y_n(t)) \)

\[
\dot{p}(t) = -H_x(p(t), x(t), u(t), \omega, t) - r(t),
\]

\[
\langle H_x(p(t), x(t), u(t), \omega, t) + r(t), y(t) - x(t) \rangle = \Delta_{y(t)} H(p(t), x(t), u(t), \omega, t)
\]

with boundary conditions

\[
p(t_1) = -\varphi_x(\lambda, x(t_1), \omega) - q,
\]

\[
\langle \varphi_x(\lambda, x(t_1), \omega) + q, y(t_1) - x(t_1) \rangle = \Delta_{y(t_1)} \varphi(\lambda, x(t_1), \omega),
\]

in which by definition we set \( r(t) = 0, q = 0 \) in the case of linearity of the functions \( f, F_0, \varphi \) with respect to \( x \) (problem (4), (5) linear by state), and also in the case of \( y(t) = x(t) \) for the corresponding \( t \in T \).

In the problem linear in the state (4), (5) the modified conjugate system (7) - (10) by definition coincides with the standard conjugate system (6).

In the non-linear problem (4), (5), the algebraic equations (8) and (10) can always be analytically resolved with respect to \( r(t) \) and \( q \) in the form of explicit or conditional formulas (perhaps not in a unique way).

Thus, the differential-algebraic conjugate system (7) - (10) can always be reduced (possibly not the only way) to a differential conjugate system with uniquely determined values \( r(t) \) and \( q \).

For the available controls \( \sigma \in \Omega, \sigma^I \in \Omega \), let \( p(t, \sigma^I, \sigma, \lambda), t \in T \) be the solution of the modified conjugate system (7) - (10) for \( x(t) = x(t, \sigma^I), y(t) = x(t, \sigma), u(t) = u^I(t), \omega = \omega^I \). The definition implies the obvious equality \( p(t, \sigma, \sigma, \lambda) = \psi(t, \sigma, \lambda), t \in T \).

Let \( P_Y \) denote the projection operator on set \( Y \in R^k \) in the Euclidean norm

\[
P_Y(z) = \arg \min_{y \in Y} (\|y - z\|), z \in R^k.
\]

The projection conditions for improving the available control \( \sigma^I \in \Omega \) with the specified projection parameter \( \alpha > 0 \) have the form

\[
u(t) = P_U(u^I(t) + \alpha(H_u(p(t, \sigma^I, \sigma, \lambda), x(t, \sigma), u^I(t), \omega^I(t), s_1(t))), t \in T,
\]
\begin{equation}
\Delta_{u(t)} H(p(t, \sigma^I, \sigma, \lambda), x(t, \sigma), u^I(t), \omega^I, t) = \langle H_u(p(t, \sigma^I, \sigma, \lambda), x(t, \sigma), u^I(t), \omega^I, t) + s_1(t), u(t) - u^I(t) \rangle, \tag{12}
\end{equation}

\begin{equation}
\omega = P_W(\omega^I + \alpha(-\varphi_\omega(\lambda, x(t_1, \sigma), \omega^I) + + \int_T H_\omega(p(t, \sigma^I, \sigma, \lambda), x(t, \sigma), u(t), \omega^I, t) dt + s_2)), \tag{13}
\end{equation}

\begin{equation}
\Delta_\omega \{-\varphi(\lambda, x(t_1, \sigma), \omega^I) + \int_T H(p(t, \sigma^I, \sigma, \lambda), x(t, \sigma), u(t), \omega^I, t) dt \} = \langle -\varphi_\omega(\lambda, x(t_1, \sigma), \omega^I) + + \int_T H_\omega(p(t, \sigma^I, \sigma, \lambda), x(t, \sigma), u(t), \omega^I, t) dt + s_2, \omega - \omega^I \rangle, \tag{14}
\end{equation}

in which in equation (12) by definition is assumed \(s_1(t) = 0\) in the case of linearity of the function \(f\), \(F_0\) by \(u\) (problem (4), (5) linear by the control \(u\)), or in the case \(u(t) = u^I(t)\) for the corresponding \(t \in T\). Similarly, in (14), by definition \(s_2 = 0\), in the case of linearity of functions \(f, F, \varphi\) by \(\omega\) (linear by the parameter \(\omega\) problem (4), (5)), and also for \(\omega = \omega^I\).

Equations (12) and (14) can always be uniquely resolved with respect to the quantities \(s_1(t)\) and \(s_2\) (perhaps not the only way).

According to [4], the solution \(\sigma = (u, \omega)\) of the system (11) - (14) provides an improvement in control \(\sigma^I \in \Omega\) for any parameter \(\alpha > 0\) with an estimate of the improvement of the functional

\begin{equation}
\Delta_\sigma L(\lambda, \sigma^I) \leq -\frac{1}{\alpha} \int_T \|u(t) - u^I(t)\|^2 dt - \frac{1}{\alpha} \|\omega - \omega^I\|^2. \tag{15}
\end{equation}

At the same time, control improvement is guaranteed not only in a sufficiently small neighborhood of the initial control \(\sigma^I \in \Omega\), i.e. the improvement procedure under consideration has the property of nonlocality, in contrast to known gradient methods and other local methods for improving control.

The conditions (11) - (14) are considered as a fixed point problem in the control space for the uniquely chosen control operator defined by the right-hand sides of these conditions.

As indicated in [4], problem (11) - (14) is an equivalent boundary value problem in the state space

\begin{align*}
\dot{x}(t) &= f(x(t), u^\alpha(t), \omega^\alpha, t), x(t_0) = x^0, \\
\dot{p}(t) &= -H_x(p(t), x^I(t), u^I(t), \omega^I, t) - r(t), \\
\langle H_x(p(t), x^I(t), u^I(t), \omega^I, t) + r(t), x(t) - x^I(t) \rangle &= \Delta_{x(t)} H(p(t), x^I(t), u^I(t), \omega^I, t) \\
p(t_1) &= -\varphi_x(\lambda, x^I(t_1), \omega^I) - q, \\
\langle \varphi_x(\lambda, x^I(t_1), \omega^I) + q, x(t_1) - x^I(t_1) \rangle &= \Delta_{x(t_1)} \varphi(\lambda, x^I(t_1), \omega^I),
\end{align*}

wherein

\begin{equation}
u^\alpha(t) = P_{\nu'}(u^I(t) + \alpha(H_u(p(t), x(t), u^I(t), \omega^I, t) + s_1(t))), t \in T,
\end{equation}
\[ \Delta \omega(t) H(p(t), x(t), u^I(t), \omega^I, t) = \langle H_\omega(p(t), x(t), u^I(t), \omega^I, t) + s_1(t), u^\alpha(t) - u^I(t) \rangle \]

\[ \omega^\alpha = P_W(\omega^I + \alpha(-\varphi_\omega(\lambda, x(t_1), \omega^I) + \int_T H_\omega(p(t), x(t), u^\alpha(t), \omega^I, t)dt + s_2)), \]

\[ \Delta \omega^\alpha \{ -\varphi(\lambda, x(t_1), \omega^I) + \int_T H(p(t), x(t), u^\alpha(t), \omega^I, t)dt \} = \langle -\varphi_\omega(\lambda, x(t_1), \omega^I) + \int_T H_\omega(p(t), x(t), u^\alpha(t), \omega^I, t)dt + s_2, \omega^\alpha - \omega^I \rangle, \]

and is indicated \( x^I(t) = x(t, \sigma^I), \ t \in T. \)

The equivalence of the boundary value problem and the fixed point problem (11) - (14) is understood in the following sense. Let the pair \( (x(t), p(t)), \ t \in T, \) be a solution of the boundary value problem. Then the control \( \sigma^\alpha = (u^\alpha, \omega^\alpha) \) is a solution of the fixed point problem (11) - (14). Conversely, suppose that the available control \( \sigma^\alpha \in \Omega \) is a solution of problem (11) - (14). Then the pair \( (x(t, \sigma^\alpha), p(t, \sigma^I, \sigma^\alpha, \lambda)), \ t \in T \) is a solution of the boundary value problem. Thus, in order to improve control \( \sigma^I \in \Omega \), it is sufficient to solve the problem of a fixed point (11) - (14) or an equivalent boundary value problem.

The solutions of the fixed point problem (11) - (14) and the equivalent boundary value problem depend on the Lagrange multiplier \( \lambda \in R \). We supplement these problems by the condition that the constraint (3) be satisfied by a choice \( \lambda \in R \). As a result, we obtain conditions for improving the admissible control \( \sigma^I \in D \) in problem (1) - (3) with an estimate

\[ \Delta_\sigma \Phi_0(\sigma^I) = \Delta_\sigma L(\lambda, \sigma^I) \leq -\frac{1}{\alpha} \int_T \|u(t) - u^I(t)\|^2 dt - \frac{1}{\alpha} \|\omega - \omega^I\|^2. \]

The proposed approach to optimizing controllable systems with constraints consists in successively solving problems of improving admissible control in the form of constructed fixed point problems of a uniquely determined control operator with the additional condition of fulfilling the constraint (3).

**Conditions for Optimal Control**

On the basis of the auxiliary Lagrange problem (4), (5), we can formulate a simple sufficient condition for the existence of an optimal solution in problem (1) - (3).

**Theorem 1.** Let the admissible control \( \sigma \in D \) be optimal in problem (4), (5) with some Lagrange multiplier \( \lambda \in R \) and \( \Phi_0(\sigma) = L(\lambda, \sigma) \). Then \( \sigma \in D \) is optimal in problem (1) - (3).

**Proof.** By virtue of the optimality \( \sigma \in D \) in problem (4), (5), we have \( \Phi_0(\sigma) = L(\lambda, \sigma) \leq L(\lambda, \tilde{\sigma}) \), \( \tilde{\sigma} \in \Omega \). Hence, for \( \tilde{\sigma} \in D \subset \Omega \), we get \( \Phi_0(\sigma) \leq L(\lambda, \tilde{\sigma}) = \Phi_0(\tilde{\sigma}). \)

To find the optimal control in the Lagrange problem (4), (5), the necessary conditions for optimality of control can be successfully used.

The necessary conditions for optimality control \( \sigma \in \Omega \) in problem (4), (5) have the following form.

**Theorem 2.** Let the available control \( \sigma \in \Omega \) be optimal in problem (4), (5). Then the conditions are fulfilled in the form of the maximum principle

\[ u(t) = \arg \max_{\tilde{u} \in U} H(\psi(t, \sigma, \lambda), x(t, \sigma), \tilde{u}, \omega, t), t \in T, \quad (16) \]
\[ \omega = \arg \max_{\omega \in \hat{W}} \left\{ -\varphi_{\omega}(x(t_1, \sigma), \omega) + \int_{T} H_{\omega}(\psi(t, \sigma, \lambda), x(t, \sigma), u(t), \omega, t) dt, \hat{\omega} \right\} \]  

(17)

**Proof.** Conditions (16), (17) are a particular case of the necessary optimality conditions obtained in [4], [6] for a general class of problems, including problem (4), (5).

Conditions (16), (17) imply weakened necessary conditions in the following forms.

**Assertion 1.** Let the available control \( \sigma \in \Omega \) be optimal in problem (4), (5). Then the conditions are satisfied in the form of the differential maximum principle

\[ u(t) = \arg \max_{\hat{u} \in \hat{U}} \langle H_u(\psi(t, \sigma, \lambda), x(t, \sigma), u(t), \omega, t), \hat{u} \rangle, t \in T, \]

\[ \omega = \arg \max_{\omega \in \hat{W}} \left\{ -\varphi_{\omega}(x(t_1, \sigma), \omega) + \int_{T} H_{\omega}(\psi(t, \sigma, \lambda), x(t, \sigma), u(t), \omega, t) dt, \hat{\omega} \right\}, \]

The specified conditions of the differential maximum principle can be represented in a known projection form with a projection parameter \( \alpha > 0 \).

**Assertion 2.** Let the available control \( \sigma \in \Omega \) be optimal in problem (4), (5). Then the necessary conditions for optimality in projection form are fulfilled

\[ u(t) = P_U(u(t) + \alpha H_u(\psi(t, \sigma, \lambda), x(t, \sigma), u(t), \omega, t)), t \in T, \]

\[ \omega = P_W(\omega + \alpha(-\varphi_{\omega}(x(t_1, \sigma), \omega) + \int_{T} H_{\omega}(\psi(t, \sigma, \lambda), x(t, \sigma), u(t), \omega, t) dt)), \]

(18)

(19)

It is important to note that in order to fulfill the differential maximum principle it suffices to verify conditions (18), (19) for at least one \( \alpha > 0 \).

The structure of the necessary optimality conditions in the projection form (18), (19) allows us to consider these conditions as a fixed-point problem for a single-valued control operator defined by the right-hand sides of these conditions. This makes it possible to apply the theory and methods of fixed points for the search for extremal controls in problem (4), (5), similarly to [4], [5], [6].

In the auxiliary problem (4), (5) there is an important connection between the necessary conditions for optimality control (18), (19) and the conditions for improving control (11) - (14), which is determined by the following statements. These statements are particular cases of the statements obtained in [4], [6] for a general class of problems, including problem (4), (5).

We denote \( \Omega^\alpha(\sigma^f) \subseteq \Omega \) – the set of the fixed points of problem (11) - (14).

**Theorem 3.** In problem (4), (5), the control \( \sigma^f \in \Omega \) satisfies the necessary optimality conditions (18), (19) if and only if there exists \( \alpha > 0 \) for which the condition is performed

\[ \sigma^f \in \Omega^\alpha(\sigma^f) \]

(20)

**Proof.** Let \( \sigma^f \in \Omega^\alpha(\sigma^f) \) be for some \( \alpha > 0 \), then \( \sigma^f \) obviously satisfies conditions (18), (19). Conversely, let \( \sigma^f \) satisfy the conditions of the differential maximum principle (18), (19), then \( \sigma^f \) is a solution of the system (11) - (14) for \( \sigma = \sigma^f \) for all \( \alpha > 0 \).
Assertion 3. (differential maximum principle on the basis of the fixed point problem). Let the control \( \sigma^I \in \Omega \) be optimal in problem \([4], (5)\). Then for some \( \alpha > 0 \) the condition \( \sigma^I \in \Omega^\alpha(\sigma^I) \) is satisfied.

From the assertions obtained, other simple statements follow in problem \([4], (5)\).

1. The fixed point problem \((11) - (14)\) is always solvable for a control satisfying the differential maximum principle.

2. In the case of non-uniqueness of the solution of the problem of a fixed point \((11) - (14)\) for a control satisfying the differential maximum principle, this control can be strictly improved by virtue of estimate \((15)\).

3. The absence of fixed points in problem \((11) - (14)\) indicates that the control is not optimal.

An estimate \((15)\) of an improvement in the functional makes it possible to obtain an intensified necessary condition for the control in problem \((4), (5)\) optimality of control in comparison with the differential maximum principle.

Theorem 4. (strengthened necessary condition for optimality of control on the basis of the fixed point problem). Let the control \( \sigma^I \in \Omega \) be optimal in problem \([4], (5)\). Then for all \( \alpha > 0 \) valid the condition

\[
\Omega^\alpha(\sigma^I) = \{ \sigma^I \}.
\] (21)

**Proof.** If for some \( \alpha > 0 \) exists \( \sigma \in \Omega^\alpha(\sigma^I), \sigma \neq \sigma^I \) then, according to the estimate \((15)\), the control \( \sigma \) strictly improves \( \sigma^I \), which contradicts the optimality of the control.

Note that in \([4], (5)\), linear by the control function \( u \), conditions \((18), (19)\) are equivalent to the maximum principle \((16), (17)\) and, accordingly, condition \((21)\) strengthens the maximum principle.

The necessary conditions for optimality of control in the form of the differential maximum principle \((20)\) and in the form of the strengthened condition \((21)\) can be successfully used to verify the control of \( \sigma^I \in \Omega \) for optimality in the problem \([4], (5)\).

**Iterative Algorithm**

Various modifications of known methods of successive approximations can be used to solve problem \((11) - (14), (3) [9]\). As an illustration, we consider an explicit iterative process for \( k \geq 0 \) with a given initial control \( u^0 \in V \) for \( k = 0 \)

\[
u^{k+1}(t) = P_U(u^I(t)) + \\
+ \alpha(H_u(p(t, u^I, u^k, \lambda), x(t, u^k), u^I(t), \omega^I, t) + s^I_1(t)) \), t \in T,
\] (22)

\[
\Delta u^k(t) H(p(t, u^I, u^k, \lambda), x(t, u^k), u^I(t), \omega^I, t) = \\
= \langle H_u(p(t, u^I, u^k, \lambda), x(t, u^k), u^I(t), \omega^I, t) + s^I_1(t), u^k(t) - u^I(t) \rangle,
\] (23)

\[
\omega^{k+1} = P_W(\omega^I + \alpha(-\varphi_\omega(\lambda, x(t_1, \sigma^k), \omega^I)) + \\
+ \int_T H_\omega(p(t, \sigma^I, \sigma^k, \lambda), x(t, \sigma^k), u^k(t), \omega^I, t)dt + s^I_2)\),
\] (24)

\[
\Delta \omega^k \{ - \varphi_\omega(\lambda, x(t_1, \sigma^k), \omega^I) + \int_T H(p(t, \sigma^I, \sigma^k, \lambda), x(t, \sigma^k), u^k(t), \omega^I, t)dt \} = \\
= \langle - \varphi_\omega(\lambda, x(t_1, \sigma^k), \omega^I) + \\
+ \int_T H_\omega(p(t, \sigma^I, \sigma^k, \lambda), x(t, \sigma^k), u^k(t), \omega^I, t)dt + s^I_2, \omega^k - \omega^I \rangle,
\] (25)
\[ \Phi_1(\sigma^{k+1}) = \varphi_1(x(t_1, \sigma^{k+1})) = 0, \]  

(26)

where the quantities \( s_1^k(t) \) and \( s_2^k \) are determined according to a given method of a uniquely determined control operator for the fixed point problem (11) - (14).

At each iteration of the process (22) - (26), the equation (26) implicitly depending by variable \( \lambda \in \mathbb{R} \) is solved. For the numerical solution of this equation, known methods can be used. The criterion for ending the calculation is the condition

\[ |\Phi_1(\sigma^{k+1})| \leq \varepsilon_1, \]

where \( \varepsilon_1 > 0 \) is a given accuracy for satisfaction of constraint (3).

Calculation of iterations on the index \( k \geq 0 \) is carried out before the first condition is fulfilled

\[ \Phi_0(\sigma^{k+1}) + \varepsilon_2 \leq \Phi_0(\sigma^T), \]

where \( \varepsilon_2 > 0 \) is a given accuracy for improvement of admissible control. In this case, a new problem (11) - (14), (3) is constructed to improve the received calculation control, considered as \( \sigma^T \), and the iterative algorithm is repeated. In this case, as the initial control approximation \( \sigma^0 \in \Omega \) for \( k = 0 \), the obtained calculation control is chosen for the iterative process (22) - (26).

If no improvement control occurs, a numerical calculation of the fixed point problem (11) - (14), (3) is carried out until the following condition is satisfied

\[ \max \{ \| u^{k+1} - u^k \|_{L(T)}, |\omega^{k+1} - \omega^k| \} \leq \varepsilon_3, \]

where \( \varepsilon_3 > 0 \) is a given accuracy for the calculation of the fixed point problem. On this, the construction and calculation of successive tasks of improving control ends.

As a result, we obtain a relaxation sequence of controls \( \sigma^k \in \Omega \) satisfying constraint (3) with a given accuracy \( \varepsilon_1 > 0 \).

To analyze the convergence conditions of the iterative process, we can apply the well-known perturbation principle in the same way as in [5]. The main convergence condition of the above iterative process is the execution of the “compression” property [9] for the operator of the right-hand side of the fixed point problem.

The convergence conditions of the relaxation sequence of controls to the optimal solution can be proved on the basis of sufficient conditions for the existence of a minimizing control sequence in problems with constraints, similar to [7].

Examples

There are considered examples of improvement admissible control and the approximate solution of the optimal control problem by the fixed point method.

Example 1. The Optimization Problem of Control Parameter with Inequality Constraint

The optimization problem with control parameter and the terminal constraint of the type of inequality is considered

\[ \dot{x}(t) = u, x(0) = 0, u \in U = [-1, 1], t \in T = [0, 1], \]

(27)
\[
\Phi_0(u) = \int_T (x^2(t) - u^2)dt \to \inf_{u \in U},
\]

In the problem is easily determined optimal solution \( u^* = -1 \) with optimal value of the objective function \( \Phi_0(u^*) = -\frac{2}{3} \).

We reduce the inequality constraint (29) to the equivalent equality constraint

\[
\Phi_1(u) = x(1) + \omega = 0, \omega \geq 0.
\]

We consider the extension problem with the vector of control parameters \( \sigma = (u, \omega) \) on the basis of the regular Lagrange function

\[
L(\lambda, \sigma) = \int_T (x^2(t) - u^2)dt + \lambda (x(1) + \omega) \to \inf_{\sigma \in \Omega, \lambda \in \mathbb{R}},
\]

\[
\Omega = U \times W, W = \{\omega : \omega \geq 0\}.
\]

For available control \( \sigma \in \Omega \) we have \( x(t, \sigma) = ut, t \in T \).

Let us set the improvement problem of admissible control \( u' = 0 \), which corresponds to the improvement problem of admissible extended control vector \( \sigma^I = (0, 0) \).

The Pontryagin’s function and the modified differential-algebraic conjugate system in the Lagrange problem (27), (30) have the following form

\[
H(p, x, u, \omega, t) = pu - x^2 + u^2,
\]

\[
\dot{p}(t) = 2x(t) - r(t), p(1) = -\lambda, \]

\[-y^2(t) + x^2(t) = (-2x(t) + r(t))(y(t) - x(t)).\]

After the change, the modified conjugate system takes the differential form

\[
\dot{p}(t) = x(t) + y(t), p(1) = -\lambda.
\]

Then we obtain a solution \( p(t, \sigma^I, \sigma) = \frac{y}{2}(t^2 - 1) - \lambda, t \in T \).

The improvement conditions for available \( \sigma^I \in \Omega \) for \( \alpha > 0 \) in the form of fixed point problem in Lagrange problem (27), (30) have the following form

\[
(u, \omega) = P_\Omega(u' + \alpha(\int_T (p(t, \sigma^I, \sigma) + 2u)dt + su'), \omega' + \alpha(-\lambda)),
\]

\[
\int_T (p(t, \sigma^I, \sigma)(u - u') + (u^2 - (u')^2)dt = (\int_T (p(t, \sigma^I, \sigma) + 2u')dt + su')(u - u').
\]

After change and addition the equality constraint, the improvement problem of control takes the form

\[
u = P_U(u' + \alpha(\int_T p(t, \sigma^I, \sigma)dt + u + u')),
\]

\[
\omega = P_W(\omega' + \alpha(-\lambda)),
\]
\[ x(1, \sigma) + \omega = 0. \]  

(35)

Calculating the integral and substituting the given control \( \sigma^I = (0, 0) \) into the problem \[ (33) - (35), \] we finally obtain the system of equations

\[ u = P_U(\alpha(\frac{2}{3}u - \lambda)); \omega = P_W(-\alpha \lambda); u + \omega = 0. \]

Analyzing the conditions \( \lambda < 0 \) and \( \lambda \geq 0 \), it is easy to obtain the following cases of existence of an admissible solution of the system for different \( \alpha > 0 \):

1) for \( 0 < \alpha < 3 \) there exists the only one admissible solution \( \sigma = (0, 0), \lambda = 0 \);
2) for \( \alpha = 3 \) there exists a set of admissible solutions \( \sigma = (3\lambda, -3\lambda), -\frac{1}{3} \leq \lambda \leq 0 \);
3) for \( \alpha > 3 \) there are two admissible solutions \( \sigma = (0, 0), \lambda = 0 \) and \( \sigma = (-1, 1), \lambda = -\frac{1}{\alpha} \).

We note that for any \( \alpha > 0 \) the control \( \sigma = (0, 0) \) is a solution of the fixed point problem \[ (31), (32) \] in the Lagrange problem for \( \lambda = 0 \). Thus, this control satisfies the differential maximum principle in the Lagrange problem for \( \lambda = 0 \) according to the above theorem \[ (3) \].

By direct calculation from the found controls it is easy to determine the control \( \sigma = (-1, 1) \) ensuring the greatest improvement in the objective function in problem \[ (27) - (29) \].

We consider new improvement problem for the received control by denoting \( \sigma^I = (-1, 1) \).

The corresponding solution of the modified conjugate system is defined in the form \[ p(t, \sigma^I, \sigma) = \frac{u-1}{t}(t^2 - 1) - \lambda, \ t \in T. \] Calculating the integral of the conjugate solution and substituting it \( \sigma^I = (-1, 1) \) into problem \[ (33) - (35), \] we obtain a system of equations

\[ u = P_U(-1 + \alpha(\frac{2}{3}u - 1) - \lambda); \omega = P_W(1 - \alpha \lambda); u + \omega = 0. \]

Analyzing, we obtain the only one solution \( \sigma = (-1, 1), \lambda = 0 \) for all \( \alpha > 0 \). The obtained control \( \sigma = (-1, 1) \) satisfies the strengthened necessary optimality condition in the Lagrange problem with \( \lambda = 0 \) and, in accordance with theorem \[ (4), \] is a pretendent for optimality.

Example 2. The Optimization Problem of Control Function with Equality Constraint

There is considered the problem of optimal control of the introduction of immunoglobulin based on the mathematical model of the immune process \[ [8]. \] In the dimensionless form, the controlled model has the form

\[
\begin{align*}
\dot{x}_1(t) &= h_1 x_1(t) - h_2 x_1(t) x_2(t) - u(t) x_1(t), \quad x_1(0) = x_1^0 > 0, \\
\dot{x}_2(t) &= h_4 (x_3(t) - x_2(t)) - h_8 x_1(t) x_2(t), \quad x_2(0) = 1, \\
\dot{x}_3(t) &= h_3 x_1(t) x_2(t) - h_5 (x_3(t) - 1), \quad x_3(0) = 1, \\
\dot{x}_4(t) &= h_6 x_1(t) - h_7 x_4(t), \quad x_4(0) = 0, \\
u(t) &\in U = [0, \bar{u}], t \in T = [0, \ t_1],
\end{align*}
\]

\[ (36) \]

\[ \Phi_0(u) = x_1(t_1) \rightarrow \inf, \]

\[ (37) \]

\[ \int_T x_4(t) dt \leq m, m > 0. \]

\[ (38) \]
Variable $x_1 = x_1(t)$ characterizes the infection (virus), variables $x_2 = x_2(t), x_3 = x_3(t)$ - protective forces of the body (antibodies, plasma cells), $x_4 = x_4(t)$ - degree of damage to the body, $h_i > 0, i = 1, 8$ - constant coefficients. The initial conditions simulate the situation of infection of the organism with a small dose of the virus $x_1^0 > 0$ at the initial time $t = 0$. The control $u(t), t \in T$ characterizes the intensity of the introduction of immunoglobulins neutralizing the virus. The control $u(t) \equiv 0, t \in T$ corresponds to the absence of treatment. In the absence of treatment, the model describes the acute form of the viral disease with recovery.

The aim is to achieve the minimum concentration of viruses by the end of the course of treatment at a given time interval with the help of the introduction of immunoglobulins, while limiting the damage to the target organ.

The values of the model coefficients and the initial dose of the virus for simulating the process under consideration in the absence of treatment have the following meanings

$$h_1 = 2, h_2 = 0.8, h_3 = 10^4, h_4 = 0.17, h_5 = 0.5,$$

$$h_6 = 10, h_7 = 0.12, h_8 = 8, m = 0.1, x_1^0 = 10^{-6}.$$

With this set of model parameters, the unit of time corresponds to one day. The maximum intensity of introduction of immunoglobulins was modeled by the value $\bar{u} = 0.5$. The time interval of the treatment was chosen equal to 20 days: $t_1 = 20$.

The introduction of restriction (38) is necessary under modeling the acute form of a viral disease, when the consequences of the organ damage cannot be neglected and one of the goals of the treatment is to limit the total body damage.

The integral condition (38) by introducing an additional variable according to the rule

$$\dot{x}_5(t) = x_4(t), x_5(0) = 0$$

was reduced to the terminal condition

$$x_5(t_1) \leq m, m > 0. \tag{39}$$

In numerical experiments with the problem (36), (37) without constraint it was established that the inequality constraint (39) in the problem is active. Therefore, the following optimal control problem with constraint-equality was considered

$$\Phi_1(u) = x_5(t_1) - m = 0 \tag{40}$$

For the numerical solution of the problem (36), (37), (40), were applied the modification of the method of nonlocal control improvement (M1) (22) - (26) and the penalty method (M2), which consisted in solving a sequence of problems without constraints with penalty functional

$$\Phi(u) = \Phi_0(u) + \gamma_s \Phi_1^2(u) \to \inf, \tag{41}$$

where $\gamma_s > 0, s \geq 1$ - sequence of assignable penalty parameters.

The calculation of penalty problems without constraints (36), (41) was carried out by the conditional gradient method [11]. A practical criterion for the completion the calculation of the penalty problem for a fixed value of the penalty parameter $\gamma_s > 0$ was the condition

$$|\Phi(u^{k+1}) - \Phi(u^k)| < \varepsilon_1 |\Phi(u^k)|, \tag{42}$$

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where \( k \geq 0 \) - iteration number of control improvement using conditional gradient method, \( \varepsilon_1 = 10^{-5} \).

If condition (42) is satisfied, then check the condition

\[
|x_5(t_1, u^{k+1}) - m| < \varepsilon_2,
\]

(43)

where \( \varepsilon_2 = 10^{-4} \). If condition (43) is not performed, then the penalty parameter \( \gamma_0 > 0 \) was recalculated according to the rule

\[
\gamma_{s+1} = \beta \gamma_s.
\]

Then a new penalty problem was calculated, in which the calculated control \( u^{k+1} \) for the conditional gradient method was chosen as the initial approximation for control improvement.

The initial value of the penalty parameter \( \gamma_0 \) for \( s = 0 \) was set to \( 10^{-10} \). The multiplier value \( \beta > 1 \) was set to 10.

The final criterion for stopping the calculation by the M2 penalty method was the simultaneous fulfillment of conditions (42) and (43).

An implicit modification of the non-local improvement control algorithm (22) - (26) was used, which for the considered problem takes the following form for \( k \geq 0 \)

\[
u^{k+1}(t) = P_U(u^I(t) + \alpha H_u(p(t, u^I, u^k), x(t, u^{k+1}), u^I(t), t), t \in T,
\]

\[x_5(t_1, u^{k+1}) - m = 0,
\]

where \( H(p, x, u, t) \) - Pontryagin’s function in the Lagrange problem with the functional

\[L(\lambda, u) = \Phi_0(u) + \lambda \Phi_1(u) \to \inf, \lambda \in R.
\]

The numerical solution of the equality constraint on each iteration with respect to the Lagrange multiplier \( \lambda \in R \) was carried out using the standard procedure “dumpol” of the Fortran Software Package [3], realizing the deformable polyhedron method. Criterion (43) was used to achieve the specified accuracy for satisfaction of the terminal constraint.

The practical criterion for stopping the calculation of the problem for implicit modification method M1 was the condition

\[|\Phi_0(u^{k+1}) - \Phi_0(u^k)| < \varepsilon_3 |\Phi_0(u^k)|,
\]

where \( \varepsilon_3 = 10^{-5} \).

As an initial approximation in both methods M1 and M2 for \( k = 0 \) was chosen the control \( u(t) \equiv 0, t \in T \).

Comparative qualitative and quantitative results of calculations are presented in Table (1).

| The method | \( \Phi_0 \) | \( |\Phi_1| \) | \( N \) | Note |
|------------|-------------|-------------|-----|------|
| M1 | \( 1.172261 \times 10^{-20} \) | \( 1.534792 \times 10^{-5} \) | 88 | \( 10^4 \) |
| M2 | \( 2.686698 \times 10^{-19} \) | \( 1.854861 \times 10^{-5} \) | 464 | \( 10^{-6} \) |

Table 1
In this table $\Phi_0$ - the calculated value of the goal functional of the problem, $|\Phi_1|$ - the module of the calculated value of the terminal functional-constraint, $N$ - the total number of calculated phase and conjugate Cauchy problems. In the notes for the penalty method M2 is specified the value of the penalty parameter, in which the given accuracy (43) of the terminal constraint is provided, and for the non-local improvement method - the value of the projection parameter $\alpha > 0$ providing convergence.

The calculation control in both methods is accurate to a day, piecewise-constant with a switching point at time $t = 5$ from the maximum value $u = 0.5$ to the minimum $u = 0$ value and backward switching at the time $t = 14$.

In the framework of the problem under consideration, the proposed non-local approach of fixed points allows achieving a significant reduction of the computational cost, measured by the total number of solved Cauchy problems, in comparison with the standard penalty method.

Conclusion

The novelty of the developed method of non-local improvement of control is to present a system of conditions for improving the admissible control in the form of a fixed point problem of a constructed control operator.

The proposed fixed point approach consists in constructing a relaxation sequence of admissible controls based on a system of control improvement conditions in an auxiliary problem without constraints with a regular Lagrange functional.

The developed form of the system for improving control in the form of a fixed point problem allows one to apply the theory and methods of fixed points to construct relaxation sequences of admissible improving controls in optimal control problems with constraints.

The constructed control improvement algorithm is characterized by the non-locality of control improvement; the absence of a laborious procedure for needle-shaped or convex variation of the control in a small neighborhood of the improved control characteristic for gradient methods; the execution of terminal constraints. These properties are essential factors for increasing the efficiency of solving nonlinear optimal control problems with constraints.

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References


