Truss Size Optimization by Using Ratio-Extremum Method: Considering Self-weight Loads, Static and Dynamic Constraints

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Abstract. For structural self-weight loads being important component in the loads, ignoring self-weight loads cannot be considered as reasonable approximation. In addition, the constraints of stresses, displacements and frequencies must be simultaneously considered to satisfy the requirements of engineering structures. According to the saddle point theory, Ratio-Extremum method is firstly proposed to solve the size optimization problem of the truss simultaneously considering self-weight loads, static and dynamic constraints. It is a great advantage that the step-size factors can be directly determined. Two cases of 72-bar special truss are used to show the effectivity and rationality.

Introduction

For heavy or precision structures, such as tower crane and high-precision antenna, structural self-weight loads are important component of the loads. Self-weight loads do greatly affect on the process and result of structural optimization, and greatly change structural configuration. These are also verified by the results of continuous topology optimization considering self-weight loads [1].

For engineering structures, the constraints of stresses, displacements and frequencies must be simultaneously considered to meet the actual needs [2]. Most of the optimization methods discussed the problems considering static and dynamic constraints, still used numerical examples of only static problems or dynamic problems [3,4,5].

To solve the optimization problems of the truss considering self-weight loads, static and dynamic constraints, it is to firstly discuss how to iteratively solve Lagrangian multipliers and cross-sectional areas, and automatically determine their step-size factors, according to the saddle point theory by using the characteristics of structural functional sensitivities.

Optimality Conditions

For the size optimization, the minimum weight of the truss under the constraints of axial stresses, nodal displacements and forbidden frequencies can be expressed as:

\[
\begin{align*}
\text{min} & \quad W(A) = \sum_{n=1}^{N} \gamma_n L_n A_n \\
\text{s.t.} & \quad g_{\sigma_n}(A) = \sigma_n / \langle \sigma_n \rangle - 1 \leq 0 \quad (n = 1, 2, \ldots, N) \\
& \quad g_{u_i}(A) = u_i / \langle u_i \rangle - 1 \leq 0 \quad (i = 1, 2, \ldots, I) \\
& \quad g_{\omega_j}(A) = 1 - \omega_j / \langle \omega_j \rangle \leq 0 \quad (j = 1, 2, \ldots, J) \\
& \quad g_{\omega_s}(A) = \omega_s / \langle \omega_s \rangle - 1 \leq 0 \quad (s = 1, 2, \ldots, S) \\
& \quad 0 \leq A^L \leq A \leq A^U
\end{align*}
\]  

(1)
Wherein, $A$ denotes the vector composed by cross-sectional areas. $W(A)$ is the objective function. $\gamma_n$ is specific weight of the $n^{th}$ bar, $A_n$ is its cross-sectional area, and $L_n$ is its length. $g_{ax}(A)$ denotes axial stress constraint of the $n^{th}$ bar, $\sigma_n$ is axial stress, and $[\sigma_n]$ is its allowable value. $g_{ul}(A)$ denotes the displacement constraint of the $i^{th}$ node, $u_i$ is the component of nodal displacement vector, $[u_i]$ is its allowable value. $g_{fl}(A)$ denotes the upper limit constraint of the $j^{th}$ frequency, $\omega_j$ is the $j^{th}$ frequency, $[\omega_j]$ is the upper limit. $g_{fl}(A)$ denotes the lower limit constraint of the $s^{th}$ frequency, $\omega_s$ is the $s^{th}$ frequency, $[\omega_s]$ is the lower limit. $A_L \leq A \leq A_U$ are the limits of cross-sectional areas; $A_L \geq 0$ means some elements can be deleted.

An essential extremum condition of Augmented Lagrangian function of Problem 1 is:

$$\nabla L(A^*, \lambda^*) = \begin{cases} 
0 & (A_m^* = A_m^U) \\
0 & (A_m^L < A_m^* < A_m^U) \\
g_m & (A_m^* = A_m^L) 
\end{cases} \quad (m = 1, 2, \ldots, N)$$

(2)

Wherein, $\lambda^*$ is the vector of Lagrangian multipliers on the optimal point $A^*$.

The dual problem of Problem 1 can be expressed as:

$$\begin{aligned} 
\text{max} & \quad \phi(\lambda) = \min_A \left[ L(A, \lambda) \right] \\
\text{s.t.} & \quad \lambda \geq 0
\end{aligned}$$

(3)

Then, its essential extremum condition is

$$\sum_{m=1}^{N} \frac{\partial L}{\partial A_m} \frac{\partial A_m}{\partial \lambda_m} + \frac{\partial L}{\partial \lambda_m} = 0$$

(4)

### Ratio-extremum Method

Step 1, solving the multipliers iteratively. The iterative solution can be written as:

$$\lambda^{(k+1)} = \lambda^{(k)} - \beta^{(k)} d_\lambda^{(k)}$$

(5)

Wherein, $\beta$ is the step-size factor, and $d_\lambda$ is its searching direction. And, $k$ denotes the $k^{th}$ step.

By using the left second term in Eq.4, the searching direction is determined by

$$d_\lambda = \begin{bmatrix} -g_1 & -g_2 & \cdots & -g_j & \cdots & -g_J \end{bmatrix}^T.$$

(6)

Wherein, $g_j$ denotes the $j^{th}$ constraint.

By using the second sub-equation of Eq.2 and Eq.4, the solution of the optimal step-size factor can be determined by [6]

$$\beta = \frac{2 d_\lambda^T d_\lambda}{d_\lambda^T G_\lambda G_\lambda d_\lambda}$$

(7)

Wherein, $G_\lambda$ denotes Jacobi matrix for partial derivatives of the constraints, and $G_\lambda$ denotes Jacobi matrix for partial derivatives of the constraints and the objective. If any of the multipliers is not greater than zero, the corresponding constraint term is set to be zero.

Step 2, solving the design-variables iteratively. The iterative solution can be written as:

$$A^{(k+1)} = A^{(k)} - \alpha^{(k)} d_p^{(k)}$$

(8)

Wherein, $\alpha$ is the step-size factor, and $d_p$ is its searching direction.

Noticed the characteristics of the truss functions [7] and affine invariance [8], by multiplying $A_m^2$ by the second sub-equation of Eq.2, and transforming it into quadratic equation with respect to $A_m$. 

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it can be derived as:

\[ A_m = h_m^g + \sqrt{(h_m^g)^2 + h_m^o}, \]  

(9)

\[ h_m^o = A_m^2 \sum_{n=1}^{N} \frac{\lambda_{mn}}{[\sigma_n]} C_n K^{-1} K^{-1} P_e \left( \gamma_m L_m + A_m \sum_{i=1}^{j} \lambda_{ui} e_i K^{-1} K^{-1} P_e \right) \left( \gamma_m L_m \right), \]

(10)

\[ h_m^g = A_m \sum_{n=1}^{N} \frac{\lambda_{mn}}{[\sigma_n]} C_n K^{-1} \left( -\frac{\partial P_e}{\partial A_m} + k_m K^{-1} P_e \right) \left( 2\gamma_m L_m \right) + A_m \sum_{j=1}^{[\omega_i]} \frac{\lambda_{uj}}{2\omega_j} \phi \left( k_m - \omega^2 M_m \right) \phi \left( 2\gamma_m L_m \right) - A_m \sum_{j=1}^{S} \frac{\lambda_{uj}}{2\omega_j} \phi \left( k_m - \omega^2 M_m \right) \phi \left( 2\gamma_m L_m \right). \]

(11)

Wherein, \( K^{-1} \) is the inverse matrix of the global stiffness matrix. And, \( k_m \) is the \( m \)th element’s unit stiffness matrix expanded to the same order as the global stiffness matrix. \( M \) is the global mass matrix of the truss. \( M_m \) is the \( m \)th element’s unit mass matrix expanded to the same order as the global mass matrix. \( P_e \) is nodal self-weight load vector. \( P_o \) is nodal external load vector. \( e_i \) is the unit row vector with the \( i \)th component being 1 and others being zero. \( C_m \) is the row vector formed by the linear combination of the unit row vectors.

If the term in the square root of Eq.9 is not negative, the components of searching direction vector can be directly determined:

\[ d_{pn} = A_m - h_m^g - \sqrt{(h_m^g)^2 + h_m^o}. \]

(12)

Otherwise, the components are determined by the third sub-equation of Eq.2, i.e.,

\[ d_{pn} = A_m - A_n^{-1}. \]

(13)

The step-size factor of cross-sectional areas is set to be:

\[ \alpha = \frac{(\nabla L_A)^T d_p}{|\nabla L_A| \cdot |d_p|}. \]

(14)

If any of cross-sectional areas is less than the lower limit, set the corresponding component of the vectors to be zero.

The above two steps will be repeated until the termination conditions occur.

**Numerical Examples**

To compare the results obtained from the proposed method with others, a 72-bar spatial truss as shown in Fig.1 with self-weight loads will be presented.

The elastic modulus \( E \) is 68.95 GPa, and the specific weight of materials \( \gamma \) is 2767.99 kg/m\(^3\). The lumped masses at 1\(^{st}\) to 4\(^{th}\) node are \( M_c = 2268 \) kg. The external loads at 1\(^{st}\) node are \( P_x = 22.25 \) KN, \( P_y = -22.25 \) KN and \( P_z = -22.25 \) KN.

The gravitational acceleration is set to be 9.85 m/s\(^2\). The self-weight loads are along negative \( y \)-direction and changing with cross-sectional areas. The allowable axial tensile and compressive stresses are 172.375 MPa and -86.18 MPa, respectively. The allowable nodal displacements are \( [u_i] = \pm 0.00635 \) m. Forbidden frequency band is 4—10Hz.
The initial cross-sectional area of the bars is set to be $4 \times 10^{-4}$ m$^2$. The lower limit is $1 \times 10^{-7}$ m$^2$, and the upper limit is $0.03$ m$^2$. The initial design point is out of the area composed of all constraints, and then the initial value of multipliers is set to be 0.

In the case with forbidden frequency band, its termination condition is to satisfy all of the constraints. In another case without forbidden frequency band, its termination condition is the relative weight between two iterations being not greater than $10^{-7}$. The optimization results are shown in Table 1.

Table 1. Comparisons with other methods for 72-bar spatial truss.

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<td>$4.529784 \times 10^{-4}$</td>
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<td>$A_{23} \cdots A_{30}$</td>
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<td>$\leq 172.375$ and $86.18$</td>
<td>$\leq 172.375$ and $\geq 86.18$</td>
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<td>Frequency [Hz]</td>
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<td>$f_i \geq 4$; $f_i \geq 6$</td>
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<td>178.676</td>
<td>437.85</td>
<td>327.691</td>
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<tr>
<td>Bar number</td>
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<td>54</td>
<td>48</td>
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<td>192</td>
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<td>192</td>
<td>N/A</td>
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</table>

Figure 1. 72-bar spatial truss.
Conclusion

According to the saddle point theory, Ratio-Extremum method for the size optimization is proposed by using the characteristics of the truss functions. The step-size factors can be directly derived and automatically calculated.

For the 72-bar spatial truss with self-weight loads, in the case with the constraints of axial stresses and nodal displacements, the proposed method only needs 192 analysis number, and the optimization weight of 178.676 kg is about 8.2% more than the weight of 165.130 kg presented by Degertekin [5] without considering self-weight loads.

Moreover, in another case with the constraints of axial stresses, nodal displacements, and forbidden frequency band of 4-10Hz, the proposed method needs 335 iterations, and the optimization weight of 186.462 kg is about 43.1% less than the weight of 327.691 kg presented by Miguel [3] without considering self-weight loads.

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References


