Gauss Pseudospectral Method Based Trajectory Optimization for Hypersonic Glide Vehicles

Jia-zhi GAO$^{1,2}$, Xiao-qian CHEN$^1$,* and Jian-ying SONG$^2$

$^1$College of Aerospace Science and Engineering, National University of Defense Technology, Changsha 410073, China

$^2$Taiyuan Satellite Launch Center, Taiyuan 030027, China

*Corresponding author

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Abstract. In order to meet the requirements of trajectory planning under complex constraints and avoid the occurrence of ballistic singularities, a set of motion models based on the idea of pole transformation is adopted, considering the influence of the earth rotation. Based on the model of pole transformation motion, the trajectory of hypersonic glide vehicle is simulated and some meaningful conclusions are obtained.

Introduction

The trajectory optimization for aircraft has always been a concern and a complex problem$^{[1-3]}$. The most significant feature of glide trajectory optimization problem distinguished from other trajectory optimization problems is that hypersonic glide vehicles fly at high speed in the near-space for a long time, and the aerodynamic and thermal environment is very bad. Therefore, how to meet the aerodynamic and thermal constraints are key problems for hypersonic glide trajectory optimization. In addition, the trajectory optimization of hypersonic glider needs to consider various constraints to meet specific flight tasks. The trajectory optimization design under such many constraints is a complex problem. In order to solve this problem, based on the flight characteristics of hypersonic gliders, appropriate strategies and optimization methods should be adopted$^{[4-5]}$.

The trajectory optimization is actually an optimal control problem$^{[6]}$. The solution can be divided into two categories: direct method and indirect method. Direct method can be further divided into direct shooting method and collocation method. Considering the complexity of the gliding trajectory optimization, the indirect method is usually difficult to meet the optimization constraints$^{[7]}$. In this paper, the adaptable direct method is adopted to solve the gliding trajectory optimization. The Gauss pseudo-spectral method is a type of direct methods developed in recent years. It has been widely used in the field of flight vehicle trajectory optimization$^{[8]}$. In this paper, the Gauss pseudo-spectral method is applied to optimize the trajectory of hypersonic gliding vehicle under complex constraints, and the maximum voyage optimization of CAV-H aircraft is taken as an example.

Dimensionless Pole-transformed Kinematics Model of Hypersonic Gliding Vehicles

The kinematics model of gliding vehicle is usually established in the half-speed coordinate system at home and abroad. When the geocentric latitude or the local trajectory angle is $\pm \pi/2$, the half-speed kinematics model will be singular. In order to avoid singularity, the pole-transformation model is proposed here. The transformed dynamic equations with time as independent variable are as following.
\[
\begin{align*}
\frac{d\hat{\sigma}}{dt} &= \frac{L\sin \nu}{V\cos \hat{\theta}} + \frac{\dot{V}}{r} \tan \hat{\phi} \cos \sin \hat{\sigma} + C_\sigma + \bar{C}_\sigma \\
\frac{d\hat{\theta}}{dt} &= \frac{L\cos \nu}{V} - \frac{g \cos \hat{\theta}}{V} + \frac{\dot{V}}{r} \cos \hat{\theta} + C_\theta + \bar{C}_\theta \\
\frac{d\hat{V}}{dt} &= -D - g \sin \hat{\theta} + \bar{C}_v
\end{align*}
\] (1)

Among them, \(g = \mu \frac{1}{r^2}\) is gravitational acceleration, \(\mu\) is gravitational constant; \(L\) and \(D\) represent lift and drag acceleration, respectively; \(\hat{r}\) is the transformed geocentric distance (the following variables are also defined in the transformed coordinate system); \(\hat{\nu}\) is the geocentric latitude; \(\hat{V}\) is velocity; \(\hat{\nu}\) is the local speed angle; \(\hat{\sigma}\) is the flight yaw angle; \(\nu\) is the bank angle; \(C_\sigma\) and \(C_\theta\) are the Coriolis acceleration terms due to Earth rotation; \(\bar{C}_\sigma\), \(\bar{C}_\theta\), and \(\bar{C}_v\) are transported acceleration terms.

Because the magnitudes of the glider’s motion variables are very different, which are not conducive to the optimization calculation, so the dimensionless treatment must be carried out. In this paper, the geocentric distance \(\hat{r}\), velocity \(\hat{V}\), time \(t\), and Earth rotation angle \(\omega_e\) are treated as follows:

\[r = \frac{\hat{r}}{R_0}, \quad V = \frac{\hat{V}}{\sqrt{g_0 R_0}}, \quad T = \frac{t}{\sqrt{R_0/g_0}}, \quad \omega_e = \frac{\omega}{\sqrt{g_0/R_0}}\] (2)

In the above formulas, \(g_0\) is the acceleration of gravity at the sea level, and \(R_0\) is Earth’s average radius.

Using formulas (2), after the non-dimensional treatment of the pole-transformed kinematics model (1), we get

\[
\begin{align*}
\frac{d\sigma}{dT} &= \frac{V}{r} \cos \theta \tan \phi \sin \sigma + \frac{L\sin \nu}{V\cos \hat{\theta}} + \bar{C}_\sigma + \bar{C}_{L} \\
\frac{d\theta}{dT} &= \frac{\hat{T}}{V} \cos \nu - \frac{\cos \theta}{r \hat{V}} + \frac{\dot{V}}{r} \cos \theta + \bar{C}_\theta + \bar{C}_{L} \\
\frac{d\hat{V}}{dT} &= -\bar{B} - \frac{\sin \theta}{r^2} + \bar{C}_v
\end{align*}
\] (3)

Among them, the dimensionless lift and drag acceleration are

\[
\begin{align*}
\bar{L} &= \frac{1}{2Mg_0} \rho (V^2 \hat{V})^2 S C_L \\
\bar{B} &= \frac{1}{2Mg_0} \rho (V^2 \hat{V})^2 S C_D
\end{align*}
\] (4)

The cordial acceleration term and the transported acceleration term after the dimensionless treatment are expressed as following

\[
\begin{align*}
\bar{C}_\sigma &= (2\sigma_\sigma - 2 \tan \theta (\dot{\sigma}_\sigma \sin \sigma + \dot{\sigma}_\sigma \cos \sigma)) \\
\bar{\sigma}_\sigma &= -\frac{\hat{r}}{V \cos \theta} (\dot{\sigma}_\sigma \sin \sigma - \sigma_\sigma \dot{\sigma}_\sigma \sin \sigma) \\
\bar{C}_\theta &= 2(\dot{\sigma}_\theta \sin \sigma - \dot{\sigma}_\theta \cos \sigma) \\
\bar{\sigma}_\theta &= \frac{\hat{T}}{V} \left[ \sigma_\sigma \dot{\sigma}_\theta \sin \sigma + \sigma_\sigma \dot{\sigma}_\theta \cos \sigma + (\dot{\sigma}_\sigma^2 \sigma_\sigma + \dot{\sigma}_\sigma \dot{\sigma}_\sigma \sin \theta) \right] \\
\bar{C}_v &= \frac{\hat{T}}{V} \left[ \dot{\sigma}_\sigma \sigma_\sigma \cos \theta \sin \sigma + \dot{\sigma}_\sigma \sigma_\sigma \cos \sigma \sin \sigma + (\dot{\sigma}_\sigma \sigma_\sigma + \dot{\sigma}_\sigma \sigma_\sigma \sin \theta) \right]
\end{align*}
\] (5)

In equations (5), \(\bar{\sigma}_\sigma\), \(\bar{\sigma}_\theta\), and \(\bar{\sigma}_v\) are expressed as following
\[
\ddot{x} = \ddot{A} (\cos \lambda \cos \phi \cos \phi_p \cos A_p \\
+ \sin \lambda \cos \phi \cos \phi_p \sin A_p + \sin \phi \sin \phi_p)
\]
\[
\ddot{y} = \ddot{A} (-\sin \lambda \cos \phi \cos A_p + \cos \lambda \cos \phi_p \sin A_p)
\]
\[
\ddot{z} = \ddot{A} (-\cos \lambda \sin \phi \cos \phi_p \cos A_p - \sin \phi \sin \phi_p)
\]

The kinematics equations after the dimensionless treatment are in the same form as the original equations, that are,

\[
\frac{d\dot{r}}{dr} = \dot{\theta} \sin \theta \\
\frac{d\lambda}{dr} = \frac{\dot{V} \cos \theta \sin \sigma}{\dot{r} \cos \phi} \\
\frac{d\phi}{dr} = \frac{\dot{V} \cos \theta \cos \sigma}{\dot{r}}
\]

Considering the dimensionless treatment, the inequality constraints can be expressed as

\[
\dot{Q} = K \rho^{0.5} (\dot{V}^2) \leq \dot{Q}_{\text{max}}
\]
\[
q = \frac{1}{2} \dot{\rho} (\dot{V}^2) \leq q_{\text{max}}
\]
\[
n = \sqrt{\dot{L}^2 + \dot{D}^2} \leq n_{\text{max}}
\]

**Trajectory Optimization using Gauss Pseudo-spectral Method**

The Gauss pseudo-spectral method uses Legendre polynomial roots as discrete points, and discrete state variables and control variables. Then Lagrange interpolation is adopted to approximate the state variables and control variables. The Gauss pseudo-spectral method converts the optimal control problem of trajectory optimization to nonlinear programming problem. In theory, the Gauss pseudo-spectral method can be used to optimize the optimal design of gliding trajectory under complex constraints.

All of the roots of the \( K \) order Legendre polynomials \( P_k(\tau) \) are chosen as the interior collocation points in the interval, noted as \( K = \{\tau_1, \ldots, \tau_K\} \). The nodes in the \( K \) are increasing monotonically in the \([-1,1]\). Taking \( K \) as interpolating nodes, the \( K \) order Lagrange interpolation of the state variables is as follows:

\[
x(\tau) \approx X(\tau) = \sum_{i=0}^{K} L_i(\tau) x(\tau_i)
\]

In eq. (8),

\[
L_i(\tau) = \prod_{j=0, j\neq i}^{K} \frac{\tau - \tau_j}{\tau_i - \tau_j}
\]

is the Lagrange interpolation basis function. Due to the nature of Lagrange interpolation, the approximation of the state are exactly equal with the true value on the interpolation point \( K \), that is, \( x(\tau_i) = X(\tau_i), i = 0, \ldots, K \).

The equations (3) and (6) are differential equations. The left sides are the derivatives of the state variables. The above state interpolation is used to approximate its differential on the collocation nodes.

\[
\dot{x}(\tau_i) \approx \dot{X}(\tau_i) = \sum_{i=0}^{K} \dot{L}_i(\tau_i) X(\tau_i) = \sum_{i=0}^{K} D_i X(\tau_i)
\]
In eq.(9), \( k = 1, \ldots, K \), \( D \in \mathbb{R}^{K \times (K+1)} \) is the differential matrix, which is not related to the optimization problem itself, but is only related to the selection of the collocation points. \( D_u \) is the differential of the Lagrange interpolation base function:

\[
D_{iu} = \hat{L}_i(\tau_i) = \begin{cases} 
\frac{(1+\tau_i)(\dot{P}_k(\tau_i) + P_k(\tau_i))}{(\tau_i - \tau_i)(1+\tau_i)P_k(\tau_i) + P_k'(\tau_i)} & i \neq k \\
\frac{(1+\tau_i)\dot{P}_k(\tau_i) + 2\dot{P}_k(\tau_i)}{2((1+\tau_i)P_k(\tau_i) + P_k'(\tau_i))} & i \neq k
\end{cases}
\]

In eq.(10), \( \tau_i, (k = 1, \ldots, K) \) are points in the collection \( K \).

By the above state differential approximation, the kinetic differential equation are configured on the collocation points as follows

\[
\sum_{i=0}^{K} D_{iu} X(\tau_i) - \frac{t_f - t_0}{2} f(X(\tau_i), U(\tau_i), \tau; t_0, t_f) = 0
\]

In eq.(11), \( k = 1, \ldots, K \).

In order to ensure the consistency of the discretion, the control variables are also discreted on the collocation points.

Finally, the performance index is processed, and the Lagrange type index is discreted by Gauss quadrature formulas. The resulted performance index can be obtained as follows

\[
J = \Phi(X_0, t_0, X_f, t_f) + \frac{t_f - t_0}{2} \sum_{i=1}^{K} \rho_i g(X_k, U_k, \tau_k; t_0, t_f)
\]

**Simulation Results and Analysis**

Taking the maximum range optimization of the CAV-H as an example\(^9\), the performance index can be expressed as \( J = \max(\lambda_f) \), \( \lambda_f \) is the longitudinal angle at the end of the trajectory.

The range constraint of angle of attack is \( [0^\circ, 20^\circ] \). The constraints of the stagnation point heat flux and dynamic pressure are 1600kW/m\(^2\) and 50kPa, respectively. Figure 1 to figure 3 are the optimization results of 20 nodes based on the Gauss pseudo-spectral method. The "GPM" is the result of Gauss pseudo-spectral Lagrange interpolation. The "NI" means the optimal trajectory obtained by integrating the motion equation based on the optimized control. As you can see from these figures, there are two main problems when the nodes are not very dense. First of all, although the control, stagnation point heat flux, and dynamic pressure meet the constraint requirements strictly at all nodes, the trajectory between some nodes exceeds the limit value, which leads to the infeasible results. Secondly, the numerical integration results are quite different from the GPM results. Even if the GPM results meet the requirements, the actual ballistcs may exceed the constraints.
In order to solve the problem caused by few nodes, an intuitive idea is to increase the number of nodes. Figure 4 to figure 6 are the optimization results of 50 nodes based on the Gauss pseudo-spectral method. It can be seen that, although the accuracy of numerical integration results is greatly improved after increasing the number of nodes, but the order of Lagrange polynomials greatly increases. On the one hand, the use of high order Lagrange can reduce the efficiency of interpolation, which seriously affects the speed of optimization. On the other hand, although using the roots of Legendre polynomials can avoid the phenomenon of "Runge" to a certain extent, however, when the Lagrange order is too high, the interpolation error may increase in some areas.

Summary

Based on the dimensionless motion model of hypersonic glider, the optimum design of maximum range trajectory is carried out by Gauss pseudo-spectral method. The pole-transformation model avoids the singular point of the ballistic equation. After the dimensionless processing of the parameters, the magnitude difference between the parameters is reduced, which is beneficial to the rapid optimization. From the simulation, we can see that the Gauss pseudo-spectral method can guarantee the constraints on all nodes, but it is difficult to ensure values between the nodes satisfy all constraints, especially for the gliding initial stage with significant jump amplitude, and gliding terminal stage with strict dynamic pressure constraint. If the nodes are not enough, the error of Lagrange interpolation will be too large and the results of the optimization are unbelievable. But if the
numbers of nodes are increased blindly, the amount of calculation can not only be increased, but the Runger phenomenon may appear.

References


