Periodic Solitary Wave Impulses in Finite-Length Granular Chains

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Abstract. The acoustic wave propagation in a chain of spheres was predicted based on the discrete dynamic equations of spheres. Generation of the strongly nonlinear solitary wave impulses with a harmonic input has been investigated in simulations. The size of spheres, the material properties of chain structure as well as the length of chain determine the intrinsic dynamics of the granular system and frequency bands. The forced responses of a chain under harmonic excitation with a given frequency reveal that the broad bandwidth solitary wave impulses with different periods can be created in a certain length chain with no or a small precompression force. The results exhibit great potential in biomedical ultrasound and NDT applications.

Introduction

Acoustic wave propagation along one-dimensional granular media has been the subject of increased interest in recent years\cite{1-5}, due to their rich properties caused by Hertzian contact between the particles and the applied static precompression force, and this allows the system to access near linear, weakly nonlinear and strongly nonlinear dynamics. In the first attempt of Nesterenko in this field, the strongly nonlinear solitary waves corresponding to negligible precompression force were predicted in an analytical solution\cite{6}. The existence of solitary waves in such a spherical chain was proved by Mackay\cite{7} and furthermore the existence criterion for solitary waves in the horizontal granular chain was given by Ji and Hong\cite{8}, both of which are based on the general existence theorem for localized traveling wave solutions on one-dimensional nonlinear lattices given by Friesecke and Wattis\cite{9}. The above results are based on the continuum approximation assumption and the infinite granular chains.

Notably, compressions and separations in fact both exist between particles in the generation of solitary waves in granular chains, in this situation, the numerical solution is generally used to directly solve the discrete dynamic equations of these media and it allows the existence of separation between the particles. The strongly nonlinear and nonsmooth behaviour in the traveling waves have been exhibited in the situation of no precompression using the discontinuous model\cite{10}.

In recent studies, the solitary wave impulses with wide broadband can be generated in finite-length granular chains using ultrasonic excitations in experiments, due to the effect of both the chain characteristics and the input signal\cite{1}. The solitary wave impulses demonstrate strongly nonlinear and nonsmooth behaviour, but possess a certain periodicity. This is different from the traditional solitary waves, which is of interest in development of new ultrasonic transducers.

In this paper, in order to investigate the precompression force range of solitary wave impulses existence, we will study the acoustic wave propagation in a chain of sphere with a harmonic input, using an algorithm of molecular dynamics simulation to solve the discrete dynamic equations of the particles directly. For a certain length chain, with increase of the precompression force, we will sort out the solitary wave impulses with different periods and analyze their characteristics.

The Theory and Mathematical Formulas

For a chain consisting of the identical spherical particles, the equation of motion of the nth sphere based on Hertzian contact was constructed as\cite{6}:
\[ m \ddot{u}_n = A(\delta_0 + u_{n-1} - u_n)^{3/2} - A(\delta_0 + u_n - u_{n+1})^{3/2} \]

Where \( m = \frac{4}{3} \pi \rho R^3 \).

The parameters of spheres are defined: \( R \) is radius; \( m \) is mass; \( \rho \) is density; \( E \) is Young’s modulus and \( \nu \) is Poisson’s ratio; \( \delta_0 \) is the distance of approach of centres of spheres under a static precompression force. \( u_n \) is the dynamic longitudinal displacement of the \( n \)th sphere. While the granular material is weakly compressed, the wave equation for a “Sonic Vacuum” chain was derived by Nesterenko [6] and shown in Eq. (2). The strongly nonlinear solitary waves are presented as a main form in the analytical solution. The particle velocity \( v \) of the solitary wave is given by Eq. (3).

\[ V_s = \frac{5v_s^2}{4c^2} \cos^4 \left( \frac{\sqrt{10}}{5a} x \right). \]

The characteristic spatial size of a soliton \( L_s \) is determined by the period of the solution described by Eq. (3), which equals

\[ L_s = \left( \frac{5a}{\sqrt{10}} \right) \pi \approx 5a. \]

The above models do not consider the reflection of boundaries and the dissipation of the system. However, the reflection effect due to boundaries is important in a finite-length chain. In recent studies, a new model including boundary conditions of two ends was constructed [1, 5]. It is assumed that a harmonic input is excited by a transducer to the first sphere and the last sphere contacts a fixed end, as illustrated in Fig.1. The dynamic equations of spherical particles are given in Eqs. (5).

![Figure 1. Illustration of generation of an acoustic propagation in finite-length chains of spheres.](image)

For the first sphere, positioned next to the transducer, the equation is:

\[ m \ddot{u}_1 = 2\sqrt{R} \left[ 2\theta_t(\delta_{0l} + u_0 - u_1)^{3/2} - \frac{\theta_m}{\sqrt{2}} (\delta_0 + u_1 - u_2)^{3/2} \right] + \lambda(\dot{u}_0 - \dot{u}_1)H(\delta_{0l} + u_0 - u_1) - \lambda(\dot{u}_1 - \dot{u}_2)H(\delta_0 + u_1 - u_2). \]  

(5a)

For the second sphere to the penultimate one, the equivalent equation of motion is:

\[ m \ddot{u}_i = \sqrt{2R} \theta_m \left[ (\delta_0 + u_{i-1} - u_i)^{3/2} - (\delta_0 + u_i - u_{i+1})^{3/2} \right] + \lambda(\dot{u}_{i-1} - \dot{u}_i)H(\delta_0 + u_{i-1} - u_i) - \lambda(\dot{u}_i - \dot{u}_{i+1})H(\delta_0 + u_i - u_{i+1}). \]  

(5b)

Finally, for the last sphere, the relevant equation is:

\[ m \ddot{u}_N = \sqrt{2R} \left[ \frac{\theta_m}{\sqrt{2}} (\delta_0 + u_{N-1} - u_N)^{3/2} - 2\theta_r(\delta_{0r} + u_N)^{3/2} \right] + \lambda(\dot{u}_{N-1} - \dot{u}_N)H(\delta_0 + u_{N-1} - u_N). \]  

(5c)

Here,

\[ \frac{1}{\theta_t} = \frac{1 - \nu_l^2}{E_l} + \frac{1 - \nu_s^2}{E_s}, \quad \theta_m = \frac{E_s}{1 - \nu_s^2}, \quad \frac{1}{\theta_r} = \frac{1 - \nu_r^2}{E_r} + \frac{1 - \nu_s^2}{E_s}. \]  

(6)
Where $E_l$ and $\nu_l$ are the Young’s modulus and Poisson ratio of the transducer, $E_r$ and $\nu_r$ are that of the fixed end, and $E_s$ and $\nu_s$ those of the spheres themselves. $\delta_{0l}$, $\delta_0$, and $\delta_{0r}$ denote the mutual approach caused by the static force between the transducer and the first sphere, between intermediate spheres, and between the last sphere and the fixed end respectively. The values of $\delta_{0l}$, $\delta_0$, and $\delta_{0r}$ are given by

$$
\delta_{0l} = \left( \frac{3F_0}{4\sqrt{\pi}R} \right)^{2/3},
\delta_0 = \left( \frac{3F_0}{2\sqrt{2\pi}R} \right)^{2/3},
\delta_{0r} = \left( \frac{3F_0}{4\sqrt{\pi}R} \right)^{2/3}.
$$

(7)

In Eqs. (5), the dissipation term is described by the damping force and a Heaviside function $H$ is incorporated to judge if the spheres are in contact. $\lambda$ is the damping coefficient. $u_0$ is the input displacement.

However, it is difficult to derive the acoustic wave equation based on the Eqs. (5) directly, but numerical algorithms can be developed to solve the velocities and displacements of individual particles at each time to construct wave propagation.

**Simulation of Generation of Solitary Wave Impulses in a Chain of Spheres**

A molecular dynamics simulation method using the Velocity Verlet algorithm was developed to solve the Eqs. 5[11]. Here we simulate a finite-length chain containing 10 spheres to observe the generation of solitary wave impulses with the increase of precompression force using the Velocity Verlet algorithm. A sinusoidal wave containing 30-cycles at 73 kHz is as input signal, with the displacement amplitude of the input being 1.0 $\mu$m. The relevant material parameters: $R$=0.5 mm; $E_s$ = 201 GPa; $\nu_s$ = 0.3; $\rho_s$ = 7833 Kg/m$^3$; $E_l$ = 201 GPa; $\nu_l$ = 0.3; $E_r$ = 2.45 GPa; $\nu_r$ = 0.35; $\lambda$ =0.32 Nsm$^{-1}$. The velocity waveform and the spectrum of the last sphere for the different precompression forces are illustrated in Fig. 2-4 respectively.

As shown in Fig. 2, while $F_0$ = 0, the strongly nonlinear solitary wave impulses with a period of 68 $\mu$s are generated, and each impulse exhibits the propagation characteristics of solitary wave, which is described by Nesterenko’s theory, i.e. Eqs. (3-4). Yang et al. [1] indicated that the observed effects result from a sum of a solitary wave traveling out from the source with a wave that reflects from the far end of the chain, also need the continuous harmonic input from boundary. The characteristic spatial size of a soliton equals closely the diameter of spheres multiply by 5. The in-phase Nonlinear Normal Mode (NNM) is the main mode of the solitary wave impulses. In calculation, we found substantial separations between spheres exist in the evolution of acoustic wave along the chain, which exhibits strong nonsmooth behavior. These factors result in the generation of the wide broadband in spectrum, which includes the subharmonic and the high order harmonic of the input frequency. The periodicity of the impulses was also exhibited in Fig. 2(b), since the minimum subhamornic 15 kHz is close to one fifth of the input frequency 73 kHz, which is the reciprocal of the period.

While the $F_0$ increases to 0.05 N, the second solitary wave impulses is generated in the simulation, as shown in Fig. 3. In this case, the period of the solitary wave impulses is 55 $\mu$s, and the corresponding minimum subharmonic is 18 kHz, which is one quarter of the input frequency. The result reveals that increasing a small precompressure can shorten the period of the impulses. Furthermore, as $F_0$ increases to 0.06 N, a new phenomenon can be observed, as shown in Fig. (4). In the front part of the wave form, the period of impulses is 55 $\mu$s, but in the part that follows, the period of impulse changes to around a half of 55 $\mu$s, which is the reciprocal of 36 kHz, i.e. the dominant frequency shown in the spectrum. In addition, the amplitude of the impulses develops into two groups. The first group keeps the approximate amplitude as that of the front part, and the amplitude of second group decreases. This phenomenon may be relevant to the energy transfer in the chain as well as the dissipation of the system.
Conclusions

The Velocity Verlet algorithm was used to solve the dynamic equations of particles in one-dimensional granular chains. The solitary wave impulses in finite-length chains were studied using a harmonic input with certain amplitude within small precompression force in simulations. The results exhibit that the period of the solitary wave impulses is an integer multiple of the period of the input signal. Another interesting result is that two periods can exist in the propagation of the solitary wave impulses in the chains.
Figure 4. The velocity waveform (a) and the spectrum (b) of the last sphere in a chain of 10 spheres with $F_0 = 0.06$ N.

References


