Numerical Model of Tidal Current and Sediment Applied in Huangmaohai Bay

Jie HE¹,² and Xin-sheng ZHAO³

¹Nanjing Hydraulic Research Institute, Nanjing, China
²State Key Laboratory of Hydrology-Water Resource and Hydraulic Engineering, Nanjing, China
³Hydrology Bureau of Yellow River Conservancy Commission, Zhengzhou, China

Keywords: Huangmaohai bay, Tidal current, Sediment, Mathematical model.

Abstract. In this paper, a 2D model for the simulation of shallow water flow by convection and diffusion over variable bottom is presented, which is based on a finite volume method over triangular unstructured grids. The format of Roe’s approximate Riemann is adopted to solve the flux terms. And the bed slope source term is treated by split in the form of the flux eigenvector. For the diffusion terms, the divergence theorem is employed to obtain the derivatives of a scalar variable on each triangular cell. Then, the process of tidal current and sediment in Huangmaohai bay is simulated, which flow pattern is similar with the actual flow. So it is present the model could be applied to simulate the complicated current structure in the water area around hydraulic construction.

Introduction

In the numerical discretization method, the shallow water equation has been experienced the Finite Difference Method, the Finite Element Method and the Finite Volume Method, etc. The Finite Volume Method is an improved Finite Difference Method with a combination of Finite Element Method, and it can be strictly satisfied to physical quantity conservation through integral of each control volume. Simulating by Finite Volume Method based on unstructured mesh, the numerical solution research of shallow water equations has gone from one-dimensional, two-dimensional to three-dimensional, in which boundary flux calculation between cells became the emphasis in solving shallow water equations. However, in shallow water, the diffusion motion is mostly not taken into account about the impact of shallow water equations. As the diffusion motion is an indispensable item of equations, and it’s an important factor in the stability of numerical solution of shallow water equations in simulating actual flow movement, therefore it’s necessary to conduct special studies of the numerical solution of shallow water equations combining with diffusion motion.

The calculation of numerical flux is a very important part of solving shallow water equations with Finite Volume Method based on unstructured mesh. There are a lot of solutions to the numerical flux, such as Roe’s approximate Riemann solver, HLL, as well as arithmetic average schemes and so on. The source terms treatment is also critical in shallow water equations, the simple treatment may lead to still water flowing over variable topography. The usual method is splitting source terms according to the decomposed form of eigenvectors of convective terms feature matrix. The diffusion motion contains partial differential quadratic term, and the surface integral is translated into line integral after integrated over the calculating element, but there is still a differential term, which is one of the main reasons that diffusion motion isn’t easy to deal with in solving shallow water equations. In this paper, the diffusive flux in the interface is calculated by average of element interface, and it’s treated similar to the Finite Element Method for the differential terms, so it can achieve numerical solution of shallow water equations with diffusion motion. Then the flow around a pillar in tank and the tidal flow around an artificial island are simulated, the result is present the model could be applied to simulate the complicated current structure in the water area around the hydraulic construction.
Governing Equations

When the size of plane is much larger than the vertical, two-dimensional shallow water equations can be used to describe the flow motion of calculated water area. Taking turbulence into account, flow governing equation can be expressed as:

\[
\frac{\partial U}{\partial t} + \nabla E = S + \nabla E^d
\]

with \( U = (d, du, dv)^T \), \( d \) is the total water depth, \( d = h + \eta \) (\( h \) is the water depth under horizontal plane, \( \eta \) is the surface fluctuation, figure 1), \( E = (F, G) \), and

\[
F = \left( \begin{array}{c} du \\ du^2 + gh^2 / 2 \\ duv \\ \end{array} \right), \quad G = \left( \begin{array}{c} dv \\ dv^2 + gh^2 / 2 \\ duv \\ \end{array} \right)
\]

where \( u \) and \( v \) are the velocity along the \( x \) and \( y \) coordinates, respectively.

The turbulend diffusion of equation is expressed as: \( E^d = (F^d, G^d) \), where

\[
F^d = \left( \begin{array}{c} \varepsilon_x, d\varepsilon u / \partial x \\ \varepsilon_x, d\varepsilon v / \partial x \\ \end{array} \right), \quad G^d = \left( \begin{array}{c} \varepsilon_y, d\varepsilon u / \partial y \\ \varepsilon_y, d\varepsilon v / \partial y \\ \end{array} \right)
\]

and \( \varepsilon_x, \varepsilon_y \) are the kinematic edd viscosity coefficients along the \( x \) and \( y \) coordinates, respectively, here isotropy is taken, then \( \varepsilon_x = \varepsilon_y = \varepsilon \). \( \varepsilon \) can be expressed as \( \varepsilon = k d U^* \), where \( U^* \) is the shear velocity, and it can be expressed as

\[
U^* = \frac{n \sqrt{g(u^2 + v^2)}}{d^{1/6}}
\]

The source term \( S \) is written as:

\[
S = S_0 + S_f = \left( \begin{array}{c} 0 \\ gd(S_{0x} + S_{fx}) + fv \\ gd(S_{0y} + S_{fy}) - fu \\ \end{array} \right)
\]

where \( S_{0x} \) and \( S_{0y} \) are bed slopes along the \( x \) and \( y \) coordinates, respectively, \( S_{0x} = -\partial z_b / x \), \( S_{0y} = -\partial z_b / y \), \( z_b \) is the bottom level, and \( S_{fx}, S_{fy} \) are the friction losses in terms of the Manning’s roughness coefficient \( n \):

\[
S_{fx} = -\frac{n^2 u^2 + v^2}{d^{4/3}}, \quad S_{fy} = -\frac{n^2 v^2 + u^2}{d^{4/3}}
\]

The superscript \( f \) is Coriolis force, \( f = 2\omega \sin \phi \), \( \omega \) is the Earth’s rotation speed, \( \phi \) is the geographical latitude.
Numerical Solution

To be convenient for calculating, computational domain is discrete using triangular grids, the variables are located herein at the geometric centers of the cells, and each grid cell represents a control volume. The (1) formula is integrated over computational domain, Gauss’s theorem applied to integral gives:

$$\frac{\partial}{\partial t} \int_{\Omega} U d\Omega + \int_{\partial\Omega} (E \cdot n) dl = \int_{\Omega} S d\Omega + \int_{\partial\Omega} (E^d \cdot n) dl$$

(6)

the integral of each unit can be written as:

$$\frac{\partial U_i}{\partial t} A_i + \int_{\partial\Omega_i} (E \cdot n - E^d \cdot n) dl = \int_{\Omega_i} S d\Omega_i$$

(7)

Where $A_i$ is the area of the cell $\Omega_i$, $l$ is the boundary of triangle, the boundary integral of each cell is approximated by a summation over the triangle edges.

$$\int_{\partial\Omega_i} (E \cdot n - E^d \cdot n) dl = \sum_{k=1}^{3} (E_k \cdot n_k - E^d_k \cdot n_k) l_k$$

(8)

Convective Numerical Flux Terms Calculation

The format of Reo’s approximate Riemann is adopted to solve the convective numerical fluxes through one edge of triangle (such as the edge $l$ between adjacent cells $i$ and $j$ in Fig.2, $R$ and $L$ are the both sides of $l$, respectively), the calculated type is as follows:

$$E \cdot n = \frac{1}{2} [(E \cdot n)_R + (E \cdot n)_L - \Delta E^*]$$

(9)

where $(E \cdot n)_R$ and $(E \cdot n)_L$ are numerical fluxes on both sides of interface, $\Delta E^*$ is the corrected value of flux on both sides and it’s written as:

$$\Delta E^* = \sum_{k=1}^{3} \alpha_k \tilde{p}_k$$

(10)
where $\lambda_k$ and $e_k (k = 1, 2, 3)$ are the eigenvalues and eigenvectors of convective characteristic matrix $A$, it satisfies $\Delta E = A\Delta U$, matrix form as follows:

$$A = \frac{\partial (E \cdot n)}{\partial U} = \begin{bmatrix}
0 & -\frac{u}{c-u}n_x - \frac{u}{c+u}n_y & 2un_x + vn_y \\
-\frac{v}{c-u}n_x + \frac{v}{c+u}n_y & v & -un_x + 2vn_y
\end{bmatrix}$$

The $\bar{u}$, $\bar{v}$ and $\bar{c}$ in matrix are calculated by mass-weighted average, they are expressed as:

$$\bar{u} = \frac{u_R \sqrt{d_R} + u_L \sqrt{d_L}}{\sqrt{d_R} + \sqrt{d_L}}, \quad \bar{v} = \frac{v_R \sqrt{d_R} + v_L \sqrt{d_L}}{\sqrt{d_R} + \sqrt{d_L}}, \quad \bar{c} = \sqrt{\frac{g(d_R + d_L)}{2}}$$

The eigenvalues of $A$ are:

$$\lambda_1 = \bar{u}n_x + \bar{v}n_y + \bar{c}, \quad \lambda_2 = \bar{u}n_x + \bar{v}n_y, \quad \lambda_3 = \bar{u}n_x + \bar{v}n_y - \bar{c}$$

with the corresponding eigenvectors are:

$$e_1 = (1, \bar{u} + \bar{c}n_x, \bar{v} + \bar{c}n_y)^T, \quad e_2 = (0, -\bar{c}n_y, \bar{v}n_y)^T, \quad e_3 = (1, \bar{u} - \bar{c}n_x, \bar{v} - \bar{c}n_y)^T$$

The corresponding coefficients of each term are respectively:

$$\alpha_{1,3} = \frac{\Delta d}{2} \pm \frac{1}{2c} \left[ \Delta (du)n_x + \Delta (dv)n_y - (\bar{u}n_x + \bar{v}n_y)\Delta d \right]$$

$$\alpha_2 = \frac{1}{c} \left( \left[ \Delta (dv) - \bar{v}\Delta d \right]n_y - \left[ \Delta (du) - \bar{u}\Delta d \right]n_x \right)$$

where $\Delta (\cdot) = (\cdot)_R - (\cdot)_L$.

**Diffusive Numerical Flux Terms Calculation**

The interface average is adopted to calculating the diffusion motion contained second derivative, and the advantages are simple treatment, improvement of calculating efficiency and maintaining accuracy.

$$E^d \cdot n = \frac{1}{2} [(E^d \cdot n)_R + (E^d \cdot n)_L]$$

where

$$E^d \cdot n = F^d \cdot n_x + G^d \cdot n_y = \begin{bmatrix}
0 & 0 \\
e_x \frac{d\bar{u}}{dx} & \frac{\bar{c}d\bar{u}}{\bar{c}dx} \\
e_y \frac{d\bar{v}}{dx} & \frac{\bar{c}d\bar{v}}{\bar{c}dx}
\end{bmatrix} \cdot n_x + \begin{bmatrix}
0 & 0 \\
e_x \frac{d\bar{u}}{dy} & \frac{\bar{c}d\bar{u}}{\bar{c}dy} \\
e_y \frac{d\bar{v}}{dy} & \frac{\bar{c}d\bar{v}}{\bar{c}dy}
\end{bmatrix} \cdot n_y$$

$$\forall \bar{c}$$
The divergence theorem is employed to obtain the derivatives of a scalar variable $c$ on a triangular cell $i$ (shown in Figure 3) as:

$$
\left( \frac{\partial c}{\partial x} \right)_i \approx \frac{1}{A_i} \int_{A_i} \frac{\partial c}{\partial x} \, dA \approx \frac{c_1 \Delta y_1 + c_2 \Delta y_2 + c_3 \Delta y_3}{A_i}
$$

$$
\left( \frac{\partial c}{\partial y} \right)_i \approx \frac{1}{A_i} \int_{A_i} \frac{\partial c}{\partial y} \, dA \approx \frac{c_1 \Delta x_1 + c_2 \Delta x_2 + c_3 \Delta x_3}{A_i}
$$

where

$$
\Delta y_1 = y_3 - y_2 \quad \Delta x_1 = x_3 - x_2 \quad c_1 = (c_1^L + c_1^R) / 2
$$

$$
\Delta y_2 = y_1 - y_3 \quad \Delta x_2 = x_1 - x_3 \quad c_2 = (c_2^L + c_2^R) / 2
$$

$$
\Delta y_3 = y_2 - y_1 \quad \Delta x_3 = x_2 - x_1 \quad c_3 = (c_3^L + c_3^R) / 2
$$

The viscous terms are calculated as:

$$
(\varepsilon d \frac{\partial u}{\partial x})_i \approx \varepsilon d_i \frac{u_1 \Delta y_1 + u_2 \Delta y_2 + u_3 \Delta y_3}{A_i}
$$

(13)

Model Applied in Actuality

From the Huanmaohai Sea falling current pattern in Fig.4, the tide current inside bay is mostly reversing current restricted by the shore boundary. The current in Huanmaohai Sea is divided into east, central and west three-ply by Dajin island, Hebao island, Damang island and other islands. The middle entrance (from Dajin island to Hebao island) is the main water way for the tidal current in Huanmaohai Sea. As the water depth in east entrance larger, the tidal current flow is larger than that in the western. The falling tidal current is divided into three parts through the middle entrance inside the bay, and the three-ply current is gathered together and upward to the top of bay. There is near 10 degree angle between the direction of west channel and falling tidal current, and the direction of east channel is basically consistent with tidal current. Compressed by Damang island and Sanjiaoshan island near KP8, the velocity of flood tide and ebb tide is bigger than other channel segments. The channel dredging will be not produced significantly impact on Huangmao bay at the hydrodynamic environment, and the current flow is adjusted along groove only in the water near channel, the depth of excavation larger, the phenomenon more significant.

The distribution of sediment concentration is showed in fig.5, which is simulated under the ebb tide in Huanmaohai Sea. The distribution of sediment concentration is present that large inside bay and small outside, large in deep groove and small inside shoal, big during ebb tide and small during flood
tide. The maximum sediment concentration is in the deep waters of the central bay during ebb-tide, for the bed load on bay waist arising due to tidal current. The sediment concentration along the west channel is more than 0.40kg/m$^3$, and the sediment concentration along the east channel is smaller relatively, however the sediment concentration is still around 0.30kg/m$^3$ in the mouth bar. So it can be seen that water and sediment environment near the east channel is more superior than that of west channel.

Figure 4. Falling Current Pattern.  
Figure 5. Sediment Distribution in Falling Current.

Conclusion

The unstructured grids can be adopted to simulate the shape and size of hydraulic construction actually, and the Finite Volume Method can improve the calculating efficiency and accuracy. During simulating the rivers around coastal engineering, diffusion motion is an indispensable item of shallow water equations, and it’s an important factor to the stability of numerical solution. The diffusive fluxes on interface is calculated by average fluxes through both sides of the interface, the differential terms after integrated are treated similar to the Finite Element Method, then the solving of diffusion motion is completed. The process of tidal current and sediment in Huangmaohai bay is simulated, and the result is present the model could be applied to simulate the flow movement of actual projects.

Acknowledgement

This research was financially supported by the Science Foundation from Nanjing Hydraulic Research Institute (Y211003).

References

