Queueing Model of Public Toilet and its Performance Analysis

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Abstract. This paper builds some public toilet analytical models based on the M/M/s queueing system. And it analyses the performance metrics of traditional public toilet (Tradition1) with $n$ cubicles in both women’s and men’s, traditional public toilet (Tradition2) with $2n$ cubicles in women’s and $n$ cubicles in men’s, and unisex public toilet by numerically simulating the queueing models. The results illustrate that the performances of unisex public toilet and Tradition2 are obviously better than the one of Tradition1. Therefore, unisex public toilet and Tradition2 can provide an idea and theoretical basis for solving the problem of the lady's queue.

Introduction

It is generally known that public toilets are of key importance for everybody, regardless of their gender, age, career, ethnic origin, physical ability or mental ability. However, with the development of economy and the improvement of life, we see a perennial phenomenon that there is a long queue in front of woman's lavatory in densely populated places, such as scenic area, supermarket, theaters, waiting lounge. Quite a few scholars investigated the reasons which caused the problem[1].

To solve the problem, people have proposed different kinds of methods. For example, the first solution involves building more and more modern toilets. The second solution is that the proportion of men’s and women’s can be adjusted from 1 to 1 to 2 or even 1 to 3. The third solution is to change women’s toilet behavior. And, the last solution involves installing permanent unisex toilets in public areas[2]. In facts, unisex toilets played a great role in Shanghai World Expo; they are widely implemented in Sweden. However, no scholars explored the performance of these solutions, especially the comparison between unisex toilet and improvement of the proportion. This paper will build some public toilet service models based on queue theory, and analyze the performance. The performance simulation illustrates that there are some interesting results.

The remainder of the paper is organized as follows. Section 2 builds the analytical models of different public toilets based on M/M/s queueing system after briefly introducing relevant theory and its applications. Section 3 presents primary performance metrics and the details of the analysis. Section 4 illustrates the numerical results obtained from the analytical model. Finally, Section 5 summarizes the findings, concludes the paper, and gives the future work.

Analytical Model Based on M/M/s Queueing System

Queueing theory, an important branch of stochastic operations research, was constructed by Erlang in 1909[3]. Now it has been widely used in communication[4-5], transportation[6], inventory[7], task scheduling[8-9], resource allocation[10], cloud computing[11-12] and many other areas, highlighting its powerful vitality. This paper will establish the models of women’s, men’s, Tradition1, Tradition2 and unisex public toilet based on M/M/s queueing system.

Fig. 1 shows the models of these two different types of public toilets, traditional and unisex public toilets. It can be seen easily that the models are four stages, arrival, queueing, service and departure. To analyze their performances, we present the following hypotheses.

1) There are $n$ cubicles in women’s, $n$ service facilities including cubicles and urinals in men’s,
shown in Fig. 1(a). And a urinal is regarded as a cubicle. Meanwhile, their services are independent of each other.

2) The arrivals of male and female customers both follow a Poisson distribution with the parameter $\lambda$, which is the arrival intensity. As a result, the total arrival intensity is equal to $2\lambda$. In other words, the arrival intensity is $2\lambda$ for unisex toilet, shown in Fig. 1(b).

3) The average service ratio $\mu$ of each facility in the women's follows a negative exponential distribution. Obviously, the average service ratio of the women’s queueing model is equal to $n\mu$.

4) The proportion of service time is 1 to 2 when men and women go to toilet. Therefore, the average service ratio of each facility in the men's is equal to $2\mu$, and the average service ratio of the men’s is equal to $2n\mu$.

5) Obviously, under the situation of equivalent service facilities, there are $2n$ cubicles in unisex toilet, the average service ratio of each facility follows a negative exponential distribution with parameter $1.5\mu$, and the average service ratio of the unisex public toilet is equal to $2n\times1.5\mu=3n\mu$.

6) Alternatively, each arrival customer will wait for an empty facility or be serviced directly.

Obviously, there is no restriction imposed on the customer resource and capacity for any kind of toilet. In other words, we shall employ M/M/s queueing system to analyze their performance. Consequently, the potential state set is $\{0, 1, 2, \ldots\}$. And the state flows of these queueing systems are shown in Fig. 2 on First Come First Service (FCFS) strategy.

**Primary Performance Metrics**

The primary performance indicators include average queue length, idle probability, average waiting time, average sojourn time and so on, similarly to other queueing systems. We firstly discuss the performance metrics of the simplest public toilet queueing system, women’s queueing system with $n$ cubicles.

Let service intensity of each cubicle in women’s and women’s queueing system be $\rho$ and $\rho_0$, respectively. That is to say, $\rho = \rho_{f1} = \frac{\lambda}{\mu}$, $\rho_0 = \rho_{f10} = \frac{\lambda}{n\mu}$. The queueing system will reach equilibrium state if $\rho_{f10} < 1$. Let the probability of $k$ ladies in the system be $P_k$. According to Fig. 2(a) and the law of K's algebraic equation[13], the following stationary distribution can be obtained.

$$\begin{align*}
\mu P_1 &= \lambda P_0, \\
k \mu P_k &= \lambda P_{k-1} & (1 \leq k \leq n), \\
n \mu P_k &= \lambda P_{k-1} & (k \geq n + 1).
\end{align*}$$

Incorporating the regularity condition $\sum_{k=0}^{\infty} P_k = 1$, the system reaches equilibrium state with the idle probability of
\[ P_{f10} = P_0 = \left( \sum_{k=0}^{n} \frac{\rho^k}{k!} + \frac{\rho^n}{n! 1 - \rho_0} \right)^{-1}. \]  

The other performance metrics, such as \( P_{f1n}, P_{f1w}, C_{f1}, F_{f1}, L_{f1}, T_{f1w}, T_{f1s} \), can be calculated and presented with \( \rho, \rho_u \) and \( P_{t10} \). \( P_{f1n} \) is the probability that there are just \( n \) ladies who are using all of the cubicles, and it can be calculated by the following formula

\[ P_{f1n} = \frac{(n\rho)^n P_{f10}}{n!}, \]  

\( P_{f1w} \) is the probability when ladies just arrive and have to wait, and it can be calculated by the following formula

\[ P_{f1w} = \frac{P_{f1n}}{1-\rho_0} = \frac{n\rho P_{f1n}}{n-\rho}. \]  

\( C_{f1w} \) is the average of ladies who are in queue, and it can be calculated by the following formula

\[ C_{f1w} = \frac{\rho \rho^n P_{f10}}{n!(1-\rho)^2} = \frac{\rho^{n+1} P_{f10}}{(n-1)(n-\rho)^2}. \]  

\( F_{f1} \) is the average of facilities which are in service. Obviously, it is equal to the service intensity of every cubicle. That is,

\[ F_{f1} = \rho. \]  

\( L_{f1} \) is the average queue length which includes the ladies in queue and ones being serviced. In other words, it is equal to the sum of \( C_{f1w} \) and \( F_{f1} \). That is,

\[ L_{f1} = C_{f1w} + F_{f1} = \frac{\rho^{n+1} P_{f10}}{(n-1)(n-\rho)^2} + \rho. \]  

\( T_{f1w} \) is the average waiting time before ladies being serviced. And \( T_{f1s} \) is the average sojourn time which includes the average waiting time \( T_{f1w} \) and the service time. According to Little formula[13],

\[ T_{f1w} = \frac{C_{f1w}}{\lambda} = \frac{\rho^n P_{f10}}{n! n \mu (1-\rho_0)^2}. \]  

\[ T_{f1s} = \frac{L_{f1}}{\lambda} = T_{f1w} + \frac{1}{\mu}. \]  

Now we discuss the performance metrics of men’s with \( n \) cubicles and unisex queueing system with \( 2n \) cubicles. We can easily obtain the service intensity \( \rho_m \) and \( \rho_u \) of each facility in men’s and unisex public toilet, service intensity \( \rho_{m0} \) and \( \rho_{u0} \) of men’s and unisex queueing system. Meanwhile, \( \rho_{f2} \) and \( \rho_{f20} \) of the women’s with \( 2n \) cubicles also can be obtained. That is,

\[ \rho_m = \frac{\lambda}{2\mu} = \frac{1}{2} \rho, \ \rho_{m0} = \frac{\lambda}{2n\mu} = \frac{1}{2} \rho_0, \]  

\[ \rho_u = \frac{2\lambda}{1.5\mu} = \frac{4}{3} \rho, \ \rho_{u0} = \frac{2\lambda}{2n \times 1.5\mu} = \frac{2}{3} \rho_0, \]  

\[ \rho_{f2} = \frac{\lambda}{\mu} = \rho, \ \rho_{f20} = \frac{\lambda}{2n\mu} = \frac{1}{2} \rho_0. \]  

Similarly, the primary performance metrics of men’s, unisex public toilet and women’s with \( 2n \) cubicles can be obtained. We can easily obtain the following performance metrics of Tradion 1 shown in Fig. 1(a).

\( P_{t10} \) is the idle probability when neither of men’s and women’s is in use. Obviously, it is the product of \( P_{f10} \) and \( P_{m0} \). That is,

\[ P_{t10} = P_{f10} \cdot P_{m0} = \left( \sum_{k=0}^{n-1} \frac{\rho^k}{k!} + \frac{\rho^n}{n! 1 - \rho_0} \right)^{-1} \left( \sum_{k=0}^{n-1} \frac{(\frac{1}{2}\rho)^k}{k!} + \frac{(\frac{1}{2}\rho)^n}{n! 2 - \rho_0} \right)^{-1}. \]
\( P_{t1n} \) is the probability when men’s and women’s are both just \( n \) customers being serviced. Obviously, it is the product of \( P_{t1n} \) and \( P_{mn} \). That is,

\[
P_{t1n} = P_{f1n} \cdot P_{mn} = \frac{(n\rho_0)^nP_{f10}}{n!} \cdot \frac{(\frac{1}{2}n\rho_0)^nP_{m0}}{n!}.
\]  
(11)

\( P_{t1w} \) is the probability when customers arrive and have to wait in the Tradition1 queueing system, which is equal to the sum of \( P_{f1w} \) and \( P_{mw} \). Namely,

\[
P_{t1w} = P_{f1w} + P_{mw} = \frac{\rho^{n+1}P_{f10}}{(n-1)!(n-\rho)^2} + \frac{1}{2} \frac{(n\rho_0)^nP_{m0}}{(n-1)(2n-\rho)^2}.
\]  
(12)

\( C_{t1w} \) is the average customers who are waiting for service in Tradition1. Obviously, it is equal to the sum but not the mean of \( C_{f1w} \) and \( C_{mw} \) because the facilities are not used each other. That is,

\[
C_{t1w} = C_{f1w} + C_{mw} = \frac{\rho^{n+1}P_{f10}}{(n-1)!(n-\rho)^2} + \frac{1}{2} \frac{(n\rho_0)^nP_{m0}}{(n-1)(n-\frac{1}{2}\rho)^2}.
\]  
(13)

\( F_{t1} \) is the average facilities which are in use in Tradition1. Obviously, it is equal to the sum of \( F_{f1} \) and \( F_{m} \). That is,

\[
F_{t1} = F_{f1} + F_{m} = \rho + 0.5\rho = 1.5\rho.
\]  
(14)

\( L_{t1} \) is the average queue length of Tradition1 queueing system. Obviously, it is the sum of \( L_{f1} \) and \( L_{m} \). That is,

\[
L_{t1} = L_{f1} + L_{m} = \frac{\rho^{n+1}P_{f10}}{(n-1)!(n-\rho)^2} + \frac{1}{2} \frac{(n\rho_0)^nP_{m0}}{(n-1)(n-\frac{1}{2}\rho)^2} + \frac{3}{2} \rho.
\]  
(15)

\( T_{t1w} \) is the average waiting time while the customers are in the Tradition1 queue, and \( T_{t1s} \) is the average sojourn time, which includes the average waiting time and the service time, of Tradition1 queueing system.

\[
T_{t1w} = \frac{1}{2} \left( T_{f1w} + T_{mw} \right) = \frac{\rho^n P_{f10}}{2n!n\mu(1-\rho_0)^2} + \frac{1}{2(n-1)!(n-2-\rho)^2}.
\]  
(16)

\[
T_{t1s} = \frac{1}{2} \left( T_{f1s} + T_{ms} \right) = \frac{1}{2} \left( T_{f1w} + T_{mw} + \frac{3}{2\mu} \right) = T_{t1w} + \frac{3}{4\mu}.
\]  
(17)

To compare unisex public toilet, we can similarly obtain the performance metrics of the Tradition2.

**Numerical Validation**

To compare the performance of different public toilet queueing systems, we simulate them numerically. To conveniently calculate, we take the average time women spend going to toilet for 90s. In other words, the average service ratio of each cubicle in women’s \( \mu \) is equal to 40 persons per hour. \( \rho_0 \) is equal to 0.9, less than 1, and all of the mentioned above queueing systems can reach equilibrium state when \( n=5 \), and \( \lambda = 180 \) persons per hour. We can obtain their performance metrics, shown in Table 1 by simulating numerically the relevant formulas above.

Table 1 illustrates that unisex public toilet can provide better performance than Tradition1, and the performances of unisex public toilet and Tradition2 are almost the same. However, the performance promotion of Tradition2 depends on increment of cubicles and space. And the average utilization rate of Tradition2 reduces from 67.5% to 45%, compared with Tradition1.
To better analyze their performances, we focus on the influence on the performance of Tradition1, Tradition2 and unisex queueing systems above when $\mu$ and $n$ are fixed and $\lambda$ is changed. The results are shown in Fig. 3.

We can obtain some interesting findings from Fig. 3. For example, Fig. 3(a) shows that the idle probability of unisex public toilet is larger than those of tradition1 and tradition2 and the latters are almost equal on the same value of $\lambda$. In addition, the idle probabilities of all the facilities in these different queueing systems take on the tendency of decrease with the increase of $\lambda$. Fig. 3(b) illustrates that $P_\alpha$ of Tradition2 hardly change, the one of Tradition1 first increases slowly and then decreases with the increase of $\lambda$ because of the first increase and then decrease tendency of Women’s1. However, the one of unisex queueing system increase dramatically. From Fig. 3(c), we can easily see that the average waiting probabilities of the three queueing systems increase with the increase of $\lambda$, and the one of Tradition1 increases vigorously while the others increase slowly. In other word, unisex queueing system and Tradition2 can decrease effectively the waiting probability $P_w$ of Tradition1. From Fig. 3(d), we can easily see that $C_w$ of Tradition1 increases first slowly and then vigorously and the others hardly change with the increase of $\lambda$. In other words, unisex queueing system and Tradition2 can decrease effectively the average waiting customers. It can be seen that the average facilities in service of Tradition1 and Tradition2 are equal according to the discussion above and Table 3. As a consequence, we think about facility utilization rate, the ratio of $F$ and facilities. Fig. 3(e) manifests that the average utilization rates of the three queueing systems all increase linearly, and those of Tradition1 and Tradition2 are max and min on the same value of $\lambda$, respectively. Fig. 3(f) shows that the average queue length $L$ of Tradition1 first increases slowly if $\lambda$ is less than 2, and then increases vigorously. This is why there is a long queue in front of women’s. However, the average queue lengths of Tradition2 and unisex public toilet present basically linear growth and they become increasingly less than the one of Tradition1 with the increase of $\lambda$. Finally, Fig. 3(g) and (h) demonstrate that the average waiting time $T_w$ and the average sojourn time $T_s$ of Tradition1 increase first slowly and then vigorously as its average queue length $L$. However, $T_w$ and $T_s$ of Tradition2 and

Table 1. Performance metrics of the queueing system when $n=5$ and $\lambda = 180$.

<table>
<thead>
<tr>
<th>Metrics</th>
<th>Women’s1</th>
<th>Women’s2</th>
<th>Men’s</th>
<th>Tradition1</th>
<th>Tradition2</th>
<th>Unisex</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_0$</td>
<td>0.0050</td>
<td>0.0111</td>
<td>0.1039</td>
<td>0.0005</td>
<td>0.0012</td>
<td>0.0024</td>
</tr>
<tr>
<td>$P_\alpha$</td>
<td>0.0762</td>
<td>0.0104</td>
<td>0.0499</td>
<td>0.0038</td>
<td>0.0005</td>
<td>0.0405</td>
</tr>
<tr>
<td>$P_w$</td>
<td>0.7625</td>
<td>0.1040</td>
<td>0.0908</td>
<td>0.8533</td>
<td>0.1949</td>
<td>0.1013</td>
</tr>
<tr>
<td>$C_w$</td>
<td>6.8624</td>
<td>0.0155</td>
<td>0.0743</td>
<td>6.9367</td>
<td>0.0898</td>
<td>0.1519</td>
</tr>
<tr>
<td>$F$</td>
<td>4.5</td>
<td>4.5</td>
<td>2.25</td>
<td>6.75</td>
<td>6.75</td>
<td>6.75</td>
</tr>
<tr>
<td>$T_w$</td>
<td>2.2875</td>
<td>0.0026</td>
<td>0.0248</td>
<td>1.1561</td>
<td>0.0137</td>
<td>0.0253</td>
</tr>
<tr>
<td>$T_s$</td>
<td>3.7875</td>
<td>1.5026</td>
<td>0.7748</td>
<td>2.2811</td>
<td>1.1387</td>
<td>1.0253</td>
</tr>
</tbody>
</table>

Figure 3. Effect on performance metrics by $\lambda$. 

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unisex queueing systems are basically unchangeable, and become increasingly less than those of Tradition1 with the increase of $\lambda$.

As a result, the performance metrics, such as $P_n$, $P_w$, $C_w$, $L$, $T_w$ and $T_s$, of Tradition2 and unisex public toilet are better than those of Tradition1. In other words, unisex public toilet and Tradition2 can improve effectively the performance of Tradition1. Compared with Tradition2, unisex public toilet need less resource, and can improve the facility utilization rate.

**Summary**

In order to improve the gender equity over the use of public facilities, promote the construction of public toilet rationalization and use, we established the queueing models of Women’s1 and men’s with $n$ cubicles, Women’s2 with $2n$ cubicles, Tradition1, Tradition2 and unisex public toilet with $2n$ cubicles based on M/M/s queueing system. Meanwhile, we investigated the performance metrics, such as average queue length, average waiting customers, average waiting probability, average waiting time and average sojourn time, of the queueing systems of Women’s1, men’s and unisex public toilet by their state flow graph. On the basis, we focused on exploring the performance metrics of Tradition1 and Tradition2.

Finally, we analyzed the performance of Tradition1, unisex public toilet, and Tradition2 by simulating them numerically. The results illustrated that the primary performance metrics, such as average queue length, average waiting customers, average waiting probability, average waiting time and average sojourn time, of unisex public toilet and Tradition2 are obviously superior to those of Tradition1. At the same time, unisex public toilet needs less resource and occupies less space than Tradition2. Although unisex toilet is the norm at home, many people prefer not to use a unisex public toilet on both safety and hygiene grounds. Consequently, this paper presents theoretical foundation to solve the problem that ladies queue to go to public toilet.

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