Optimal Sliding Mode Control for Nonlinear Systems with Matched and Mismatched Uncertainties Based on ENDOB

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ABSTRACT

This paper proposes a novel result on optimal sliding mode control (OSMC) approach for general nth order nonlinear systems with matched and mismatched uncertainties based on extended nonlinear disturbance observer (ENDOB). The combination of the control Lyapunov function (CLF) method with the ISMC based on disturbance observer guarantee the robustness, maintain the nearly optimal performance of the sliding-mode dynamics. In the first step, the nonlinear optimal control strategy based on the CLF is proposed for the known part of nonlinear system. As the optimal controller is highly sensitive to the system uncertainty, a ENDOB is designed to approximate matched and mismatched uncertainties. Then by designing a novel sliding mode manifold integrated with a disturbance estimation technique, a ENDOB-based OSMC method is designed for these systems. A rigorous stability analysis of the composite closed-loop system is provided using Lyapunov theory. The final two simulation results are provided to verify the effectiveness of the proposed control method.

KEYWORDS
Optimal sliding mode control (OSMC), Control Lyapunov function (CLF), Extended nonlinear disturbance observer (ENDOB), Matched and mismatched uncertainties.

INTRODUCTION

It is known that uncertainties and disturbances are always inevitable in most of practical systems, which exist in the same channel of the control input is called matched uncertainties and disturbances, otherwise, it is called mismatched uncertainties and disturbances. In recent years, the control methods of systems affected by uncertainties and disturbances have been favored by researchers [1-6]. Compared to other control methods, sliding mode control (SMC) has attracted a significant interest as a successful control strategy for nonlinear uncertain systems due to its conceptual simplicity, easy implementation, and robustness to external disturbances and model uncertainties [7-9]. Once the system states are onto the sliding manifold, the conventional SMC shows insensitivity to matched uncertainties and disturbances.

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While many practical systems such as power systems [10], electronic systems [11] and motor systems [12], are affected by mismatched uncertainties. The sliding motion of conventional SMC is severely affected by the mismatched uncertainty and disturbances, the robustness property of it no longer holds. Algorithms like LMI-based control [13-14], adaptive-based control [15-16], backstopping-based control [17], and integral sliding-mode control [18-19] are proposed to handle mismatched uncertainties in a robust way, but the price is that the nominal control performance is compromised.

As a practical alternative approach, disturbance-observer based control has been proven to be promising and effective in compensating the effects of unknown external disturbances and model uncertainties in control systems as well as will not deteriorate the existing controller [20-21]. It could completely remove the non-vanishing disturbances from system as long as they can be estimated accurately [22]. The magnitude of the discontinuous item may be reduced in SMC control and the chatter is likely to mitigate by this combination. Recently, several authors introduced a disturbance observer (DOB) for SMC to alleviate the chattering problem and retain its nominal control performance [23–26]. The idea is to construct the control law by combining the SMC feedback with the disturbance estimation based-feed forward compensation straightforwardly. However, these methods given in [23–26] are only available for the matched uncertain systems. A nonlinear extension of DO has been proposed by W. H. Chen, which estimates matched as well as mismatched disturbances [27–29]. A novel sliding-mode control based on the disturbance estimation by a nonlinear disturbance observer (NDOB) based SMC in [12] is only proposed to deal with mismatched uncertainties, it can ensure the system performance and reduce the chattering. [11] investigates an extended state observer (ESO)-based sliding mode control (SMC) approach for pulse width modulation-based DC–DC buck converter systems subject to mismatched disturbances, the proposed method obtains a better disturbance rejection ability even the disturbances do not satisfy the so-called matching condition.

Traditional SMC usually demands high control input that increases the cost of the nonlinear system. In order to minimize the control input, SMC is integrated with the optimal controller, it is called optimal sliding mode control [30-31]. The optimal control is devised for the nominal part of the system neglecting the uncertainties and SMC considered the uncertain part of the system. The OSMC was designed for linear systems as same as nonlinear system affected by the matched uncertainty [32]. The design of optimal controller for nonlinear systems, control Lyapunov function (CLF) has been effectively used. Unlike the Lyapunov function, the CLF is defined for systems with inputs having no specified feedback law. if it is possible to find the CLF for a nonlinear system, it is also possible to find the optimal control law without actually solving the HJB equation [33–34].

In this paper, aiming to improve the performance of the nonlinear system affected by matched and mismatched uncertainties, a novel nonlinear control scheme is proposed, where the optimal sliding mode control scheme is integrated with ENDOB. The optimal controller is designed based on control Lyapunov function (CLF). By fully taking into account the estimation value of disturbances, a new sliding-mode surface is designed which is insensitive to not only matched disturbances but also mismatched ones. In this paper, the contributions are listed as follows:

i) The control is proposed for a general n order nonlinear system with an enlarged class of matched and mismatched uncertainties.
ii) The combination of optimal control with the ISMC based on disturbance observer guarantee the robustness, maintain the nearly optimal performance of the sliding-mode dynamics, minimize the control input while at the same time reduce chattering.

The rest of the paper is organized as follows: the problem of conventional sliding mode controller design with mismatched and matched uncertainties for a class of nonlinear system is stated in Section 2. For tackling the matched and mismatched uncertainties, the extended nonlinear disturbance observer based integral sliding mode control combined with the optimal controller is derived in Section 3. Section 4 gives generalization controller for nonlinear system of order n and the stability analysis. The simulation results are presented in Section 5, followed by some concluding remarks in Section 6.

PROBLEM STATEMENT

Consider a class of single-input single-output dynamic nonlinear system with matched and mismatched uncertainties, depicted by

\[
\begin{align*}
\begin{cases}
\dot{x}_i &= x_{i+1} + d_i(t) \\
\quad &
\vdots \\
\dot{x}_{n-1} &= a(x) + b(x)u + d_n(t) \\
y &= x_1
\end{cases}
\end{align*}
\]

(1)

where \( x = [x_1 \cdots x_n]^T \) is the state vector, \( u \) is the control input, \( y \) is the output, \( d_i(t) \) and \( d_n(t) \) are the mismatched and matched uncertainties, respectively. \( a(x) \) and \( b(x) \) denote smooth nominal functions.

Taking a second-order system as an illustration (i.e., \( n = 2 \) for system (1)), we can get the control model of the system:

\[
\begin{align*}
\begin{cases}
\dot{x}_1 &= x_2 + d_1(t) \\
\dot{x}_2 &= a(x) + b(x)u + d_2(t) \\
y &= x_1
\end{cases}
\end{align*}
\]

(2)

**Assumption 1:** The lumped disturbances \( d_1(t), d_2(t) \) in system (2) are bounded and defined by \( d^* = \sup_{t>0} |k_1d_1(t) + d_2(t)| \).

The sliding mode surface and control law of the traditional SMC is usually designed as follows:

\[
s_i = x_2 + k_1x_1
\]

(3)

\[
u = -b^{-1}(x)[k_1x_2 + \eta_1 \text{sgn}(s_i) + a(x)]
\]

(4)

where \( k_1 > 0 \) is a design constant, \( \eta_1 > 0 \) is the switching gain to be designed. Taking the derivative of (3), and combining (2) and (4) gives:
\[ \dot{s}_1 = -\eta_1 \text{sgn}(s_1) + k_i d_i(t) + d_2(t) \tag{5} \]

Equation (5) can be concluded that the states of system (2) will reach the sliding mode surface \( s_1 = 0 \) in finite time if the switching gain \( \eta_1 \) in the control law (4) is devised such that \( \eta_1 > d^* \). Once the nominal sliding surface \( s_1 = 0 \) is reached, the sliding motion is obtained and given by:

\[ \dot{x}_i + k_i x_i = d_i(t) \tag{6} \]

**Remark 1:** Equation (6) implies that if the \( d_i(t) = 0 \), the system states can be driven to the desired equilibrium point, which implies that the conventional SMC is insensitive to matched disturbance. However, in the existence of mismatched disturbance \( d_i(t) \neq 0 \), the system state \( x_i \) is affected by the mismatched disturbance and does not converge to zero although the control law can force the system states to reach the sliding-mode surface. To this end, it is an essential reason why the nominal SMC design is only insensitive to matched disturbance but sensitive to mismatched disturbances.

**MAIN RESULTS**

In this section, the matched and mismatched disturbance rejection problem is considered for the system (2). The objective of the proposed method is to design an optimal sliding mode controller for a nonlinear system affected by matched and mismatched uncertainties. The design of observer based OSMC is divided into three divisions. (1) designing the optimal controller for a known nonlinear system, (2) estimating the uncertainties by introducing the extended nonlinear disturbance observer, (3) proposing OSMC based on ENDO for nonlinear system with matched and mismatched uncertainties. So, control \( u \) can be defined as:

\[ u = u_1 + u_2 \tag{7} \]

where \( u_i \) is designed using optimal control methodology to stabilize known nonlinear part of the system and SMC is defined as \( u_2 \).

**Assumption 2:** The disturbance \( d_i(t) \) is continuous and satisfies:

\[ \left| \frac{d_i^j(t)}{dt} \right| \leq \omega_j \quad \text{for} \quad \begin{cases} i = 1, 2, \ldots, n \\ j = 0, 1, 2, \ldots, r \end{cases} \tag{8} \]

where \( \omega_j \) is a positive number.

**Remark 2:** The class of disturbances considered here relax restrictions compared with [12] where it is assumed that \( \lim_{t \to \infty} \dot{d}(t) = 0 \).
CLF Based Optimal Controller Design

The system (1) can be converted into a class of nonlinear system with uncertainties and external disturbance:

\[
\begin{align*}
\dot{x} &= f(x) + g_1(x)u + g_2(x)d(t) \\
y &= x_1
\end{align*}
\]  

(9)

where \(x \in \mathbb{R}^n\) is the state vector, \(u \in \mathbb{R}\) denotes the control input. \(f(x), g_1(x)\) are sufficiently smooth known vector fields. \(d(t)\) represents the system uncertainties. The disturbance is not satisfy the matching condition as \(g_2(x)\) is not in the range space of \(g_1(x)\).

Neglecting the uncertainties and external disturbance, the system defined in (9) is described as:

\[
\dot{x} = f(x) + g_1(x)u_i
\]  

(10)

The performance index \(J\) chosen to optimize the control input \(u_i\) is defined as:

\[
J = \int_0^\infty x^T Qx + u_i^T R u_i \, dt
\]  

(11)

where \(Q\) and \(R\) are the symmetric positive-definite matrices, \((f, x^T Qx)\) is zero state detectable.

In order to find the optimal stabilizing controller for a nonlinear system using traditional Lyapunov functions, the Hamilton–Jacobi–Bellman equation should be solved. The equation is a nonlinear equation, so it is not easy to find an analytical expression for the solution. However, the control Lyapunov function is used as a substitute for the value function in the HJB approach to the optimal control problem without solving the HJB equation. The idea of the CFL-based controller is to define a Lyapunov candidate for the open-loop system and then design a feedback loop that makes the Lyapunov derivative negative. Here \(V(x)\) is a Lyapunov function and derivative of \(V(x)\) is:

\[
\dot{V}(x) = L_f V(x) + L_g V(x)u_i(t)
\]  

(12)

where \(L\) represents the Lie derivative operator.

Now, \(V(x)\) is a CLF if \(\forall x(t) \neq 0\),

\[
L_g V(x) = 0 \Rightarrow L_f V(x) < 0
\]  

(13)

In [30] suggested feedback controller based on CLF is defined as:
\[
\begin{align*}
 u_i = 
\begin{cases} 
 0 & \text{for } b(\chi) = 0 \\
 \frac{a(\chi) + \sqrt{a(\chi)^2 + x^T \Omega b(\chi) R^{-1} b(\chi)}}{b(\chi) b(\chi)^T} b(\chi)^T & \text{for } b(\chi) \neq 0
\end{cases} 
\end{align*}
\]

where \( a(\chi) = L_r V(\chi) \) and \( b(\chi) = L_g V(\chi) \).

**Lemma 1[30]:** The control law \( u_i \) defined in (14) stabilizes the nominal nonlinear system defined in (10) by minimizing the performance index (11).

**Remark 3:** The level curves of a CLF fully agree in shape with the value function, the formula (14) gives the real optimal control. For an arbitrary CLF, though in most cases those level curves may not fully agree, this approach based on the formula (14) will result in a suboptimal controller.

The CLF based optimal controller is usually based on the precise mathematical model of the system under consideration. But if the system is affected by uncertainties during operation, the optimal controller is most likely to fail. The performance criterion deviates from the optimal value and may even drive the system towards instability. An efficient way to inhibit this limitation is to integrate the optimal controller with the sliding mode control based on disturbance observer to ensure robustness.

**Sliding Mode Control Based on Disturbance Observer**

**A. Second Order Extended NDOB**

First, a second order ENDOB for system (2) is proposed as follows:

\[
\begin{align*}
 \dot{\hat{d}}_1 &= p_{11} + l_{11} x_1 \\
 \dot{\hat{p}}_{11} &= -l_{11} (x_2 + \hat{d}_1) + \hat{d}_1 \\
 \dot{\hat{d}}_2 &= p_{21} + l_{12} x_1 \\
 \dot{\hat{p}}_{21} &= -l_{12} (a(x) + b(x) u + \hat{d}_2) + \hat{d}_2
\end{align*}
\]

where \( \hat{d}_1, \hat{d}_2, \hat{\hat{d}}_1, \hat{\hat{d}}_2 \) are estimates of \( d_1(t), d_2(t) \) and \( \dot{d}_1(t), \dot{d}_2(t) \) respectively, \( p_{11}, p_{12}, p_{21}, p_{22} \) and \( l_{11}, l_{12}, l_{21}, l_{22} \) are auxiliary variables, \( l_{11}, l_{12}, l_{21} \) and \( l_{22} \) are user chosen constants. The stability of the extended NDOB is discussed in Section 4.

**B. Novel SMC Control with ENDOB of Second-Order System**

Combining the optimal controller with the Integral sliding mode control (ISMC) is proposed. Conventional SMC cannot tackle the mismatched uncertainty. So, a novel sliding mode manifold for system (2) under matched and mismatched disturbance is designed as:

\[
s_2 = x_2 + k x_1 - G \int_0^t \dot{x}_{nom} dt + \dot{\hat{d}}_1(t)
\]
where $G$ is the design chosen parameter matrix $[0 \ 1]$, $x_{nom}$ is the nominal part of the uncertain system (2) neglecting uncertainties and external disturbance. Hence,

$$
\dot{s}_2 = \dot{x}_2 + k_2 \dot{x}_1 - G \dot{x}_{nom} + \dot{d}_1(t) \quad (17)
$$

Then, by using (2) and (10), it follows that

$$
\dot{s}_2 = a(x) + b(x)(u_i + u_2) + k_2 \dot{x}_1 \\
- a(x) - b(x)u_i + \dot{d}_1 \\
= b(x)u_2 + k_2 x_2 + d_1 + d_2 + \dot{d}_1 \quad (18)
$$

Theorem 1: Considering the above system (2) with matched and mismatched disturbances, sliding-mode surface (16) is proposed, if the control law is designed as (7), where $u_i$ is defined as (14), control law $u_2$ is defined as

$$
u_2 = -b^{-1}(x) \left\{ k_2 \left[ x_2 + \dot{d}_1(t) \right] + \eta_2 \text{sgn}(s_2) \left[ + d_2(t) + \dot{d}_1(t) \right] \right\} \quad (19)
$$

Suppose the second-order system satisfies assumptions 2, if the switching gain is chosen such that $\eta_2 > \sup_{t>0} \left| k_2 e_{d1} + e_{d2} \right|$ and $k_2 > 0$, then the closed-loop system is asymptotically stable.

Proof:
Consider a candidate Lyapunov function as

$$V_1 = \frac{1}{2} s_2^T s_2 \quad (20)
$$

Taking derivative of $V$ in (14), we obtain that

$$\dot{V}_1 = s_2 \dot{s}_2 = s_2 (b(x)u_2 + k_2 x_2 + d_1 + d_2 + \dot{d}_1) \quad (21)
$$

Substituting (19) into (21) yields

$$\dot{V}_1 = s_2 \left\{ -k_2 \left[ x_2 + \dot{d}_1(t) \right] - \eta_2 \text{sgn}(s_2) \left[ - \dot{d}_1(t) - \dot{d}_1(t) - d_2(t) \right] \right\} \\
= s_2 \left\{ -k_2 \dot{d}_1(t) - \eta_2 \text{sgn}(s_2) \left[ - \dot{d}_1(t) + d_2(t) \right] + k_2 \dot{d}_1(t) \right\} \\
= s_2 \left\{ -\eta_2 \text{sgn}(s_2) + k_2 e_{d1} + e_{d2} \right\} \\
\leq \left[ -\eta_2 + k_2 e_{d1} + e_{d2} \right] s_2 \leq -\sqrt{2} \left[ \eta_2 - (k_2 e_{d1} + e_{d2}) \right] V_1^{\frac{1}{2}} \quad (22)$$
where \( e_{d1} = d_1 - \hat{d}_1, e_{d2} = d_2 - \hat{d}_2 \). It can be derived from \( \dot{V}_i \) (22) that the system states will reach the defined sliding surface \( s_2 = 0 \) when \( \eta_2 > \sup_{t \geq 0} \left| k_2 e_{d1} + e_{d2} \right| \), then the closed-loop system is asymptotically stable.

**Remark 4:** Since the matched and mismatched disturbance has been precisely estimated by the ENDOB, the switching gain of the proposed method can be designed much smaller than those of the traditional SMC, because the magnitude of the estimation error \( e_d \) is much smaller than the magnitude of the disturbance \( d \). The chattering problem can be alleviated to some extent in the case of the disturbance \( d \).

C. A third-order system case

Consider the following third-order system, depicted by:

\[
\begin{align*}
\dot{x}_1 &= x_2 + d_1(t) \\
\dot{x}_2 &= x_3 + d_2(t) \\
\dot{x}_3 &= a(x) + b(x)u + d_3(t) \\
y &= x_1
\end{align*}
\]

(23)

A sliding mode manifold for system (23) is designed as:

\[
s_3 = x_3 + x_2 + k_3 x_1 - G \int_0^t \dot{x}_{nom} dt + \dot{d}_1(t) + \dot{d}_2(t) + \dot{d}_3(t)
\]

(24)

**Theorem 2:** Considering the above system (23) with matched and mismatched disturbances, sliding-mode surface (24) is proposed, if the control law is designed as (7), where \( u_i \) is defined as (14), control law \( u_2 \) is defined as

\[
u_2 = -b^{-1}(x) \left\{ k_3 \left[ x_2 + \dot{d}_1(t) \right] + \eta_3 \text{sgn}(x_3) + x_3 \right. \\
\left. + \dot{d}_2(t) + \dot{d}_3(t) + \dot{x}_1(t) + \dot{x}_2(t) + \dot{x}_3(t) \right\}
\]

(25)

Suppose the third-order system satisfies assumptions 2, if the switching gain is chosen such that \( k_j > 0 \) and \( \eta_3 > \sup_{t \geq 0} \left| k_3 e_{d1} + e_{d2} + e_{d3} \right| \), then the closed-loop system is asymptotically stable.

**GENERALIZATION TO SYSTEMS OF ORDER N AND STABILITY**

**Generalization of ENDOB**

To estimate the disturbance \( d_i \) and its derivatives in the \( i \)th channel, define:

\[
\hat{d}_i^{(j-1)} = p_{ij} + l_{ij} x_i \quad i = 1, 2 \ldots, (n - 1)
\]

(26)

where auxiliary variables are defined as:
\[
\dot{y}_j = -l_{ij}(x_{i+1} + \hat{d}_i) + \hat{d}^{(j)}_i, \quad j = 1, 2 \cdots (r-1)
\]
\[
\dot{y}_r = -l_{ir}(x_{i+1} + \hat{d}_i)
\]  
(27)

The disturbance \(d_n\) and its derivatives can be estimated as:
\[
\hat{d}^{(j-1)}_n = p_{nj} + l_{nj}x_n
\]  
(28)

where auxiliary variables are defined as:
\[
\dot{p}_{nj} = -l_{nj}(a(x) + b(x)u + \hat{d}_n) + \hat{d}^{(j)}_n, \quad j = 1, 2 \cdots (r-1)
\]
\[
\dot{p}_{nr} = -l_{nr}(a(x) + b(x)u + \hat{d}_n)
\]  
(29)

**Stability of ENDOB**

Let the estimation errors be defined as:
\[
\tilde{e}_i = \left[ \begin{array}{c}
\tilde{d}_i \\
\tilde{d}_i
\end{array} \right]
\]  
(30)

where \(\tilde{d}_i = d_i - \hat{d}_i\), \(\tilde{d}_i = d_i - \hat{d}_i, \quad i = 1, 2 \cdots n\). \(\tilde{d}_i\) denotes the error in the estimation of \(d_i\) and \(\tilde{d}_i\) denotes the error in the estimation of \(\dot{d}_i\).
\[
\dot{\tilde{d}}_i = l_{ii}\tilde{d}_i - \hat{d}_i
\]  
(31)

Subtracting both sides of (31) from \(\dot{d}_i\)
\[
\dot{d}_i = -l_{i1}\tilde{d}_i + \dot{d}_i - \dot{\tilde{d}}_i
\]
\[
= -l_{i1}\tilde{d}_i + \dot{d}_i
\]  
(32)

It can be obtained that:
\[
\dot{\tilde{d}}_i = -l_{i2}\tilde{d}_i + \ddot{d}_i
\]  
(33)

Differentiating (32) and using (33):
\[
\ddot{d}_i = -l_{i1}\dot{d}_i - l_{i2}\ddot{d}_i + \dddot{d}_i
\]  
(34)

The observer error dynamics can be expressed in compact form as:
\[
\hat{e}_i = L_i \hat{e}_i + I_i d_i^{(m)}
\]  
\[
L_i = \begin{bmatrix}
-I_{i_1} & 1 & 0 & \cdots & 0 \\
-I_{i_2} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-I_{i_{n-1}} & 0 & 0 & \cdots & 1 \\
-I_{i_n} & 0 & 0 & \cdots & 0 \\
\end{bmatrix}, \quad I_i = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1 \\
\end{bmatrix}
\]

The gains \( I_i \) of the ENDOB for the disturbances \( d_i \) can always be chosen so that the eigenvalues of each \( L_i \) are in LHP.

Therefore, it is always possible to find a positive definite matrix \( P_i \) such that
\[
L_i^T P_i + P_i L_i = -Q_i
\]

where \( Q_i \) is positive definite matrix. Let \( \chi_m \) denote the smallest eigenvalue of \( Q_i \).

Defining a Lyapunov function
\[
V(\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n) = \sum_{i=1}^{n} \hat{e}_i^T P_i \hat{e}_i
\]

Derivative of \( \dot{V}(\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n) \) along (37) is found as:
\[
\dot{V}(\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n) = \sum_{i=1}^{n} \left[ \hat{e}_i^T (L_i^T P_i + P_i L_i) \hat{e}_i + 2 \hat{e}_i^T P_i d_i^{(m)} \right]
\leq \sum_{i=1}^{n} \left[ -\hat{e}_i^T Q_i \hat{e}_i + 2 \left\| P_i \right\| \left\| \hat{e}_i \right\| \left\| \omega_i \right\| \right]
\leq \sum_{i=1}^{n} \left[ -\chi_m \left\| \hat{e}_i \right\|^2 + 2 \left\| P_i \right\| \left\| \hat{e}_i \right\| \left\| \omega_i \right\| \right]
\leq \sum_{i=1}^{n} \left[ -\left\| \hat{e}_i \right\| (\chi_m \left\| \hat{e}_i \right\| - 2 \left\| P_i \right\| \left\| \omega_i \right\|) \right]
\]

Therefore, after a sufficiently long time, the norm of the estimation error is bounded by:
\[
\left\| \hat{e}_i \right\| \leq \frac{2 \left\| P_i \right\| \left\| \omega_i \right\|}{\chi_m}
\]

Let
\[
\chi_i = \max \left[ \frac{2 \left\| P_i \right\| \left\| \omega_i \right\|}{\chi_m} \right]
\]

Therefore \( \left\| \hat{e}_i \right\| \leq \chi_i \) for all \( i \).
ENOB Based SMC to Systems of Order n and stability

A sliding mode manifold for system (1) is designed as:

\[ s_n = \sum_{i=2}^{n} x_i + k_n x_1 - G \int_{0}^{t} \dot{x}_{\text{nom}} \, dt + \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \hat{d}_i^{(j-1)}(t) \]  \hspace{1cm} (41)

where \( G \) is the design chosen parameter matrix \([0 \ldots 0 \ 1]\).

**Theorem 3:** Considering the above system (1) with matched and mismatched disturbances, sliding-mode surface (41) is proposed, if the control law is designed as (7), where \( u_1 \) is defined as (14), control law \( u_2 \) is defined as

\[ u_2 = -b^{-1}(x) \left\{ k_n \left[ x_2 + \hat{d}_1(t) \right] + \eta_n sgn(s_n) + \sum_{i=3}^{n} x_i \right. \\
\left. + \sum_{i=2}^{n} \hat{d}_i(t) + \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \hat{d}_i^{(j)}(t) \right\} \]  \hspace{1cm} (42)

Suppose the general high-order system satisfies Assumptions 2, if the switching gain is chosen such that \( \eta_n > \sup_{t>0} \left| k_n e_{d1} + \sum_{i=2}^{n} e_{di} \right| \) and \( k_n > 0 \), then the closed-loop system is asymptotically stable.

**Proof:**

Consider a candidate Lyapunov function as

\[ V_n = \frac{1}{2} s_n^T s_n \]  \hspace{1cm} (43)

Taking derivative of \( V \) in (43), we obtain that

\[ \dot{V}_n = s_n^T \dot{s}_n = s_n^T (\sum_{i=2}^{n} \dot{x}_i + k_n \dot{x}_1 - \alpha \dot{x}_{\text{nom}}^T + \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \dot{d}_i^{(j-1)}(t)) \]

\[ = s_n^T \left[ x_1 + d_2 + x_3 + d_3 + \cdots + x_n + d_{n-1} + a(x) + b(x)(u_1 + u_2) \right] \]

\[ + \dot{d}_n + x_2 + k_n x_1 + a(x) - b(x)u_1 + \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \dot{d}_i^{(j)}(t) \]  \hspace{1cm} (44)

Substituting (42) into (44) yields

\[ 224 \]
\[
\dot{V}_n = s_n 
\begin{bmatrix}
    x_1 + d_2 + x_1 + d_3 + \cdots + d_n \\
    -b(x)\times b^{-1}(x) \left\{ k_n [x_2 + d_i(t)] + \eta_n \sgn(n(t)) \right\} \\
    + x_n + \sum_{i=2}^{n} d_i(t) + \sum_{j=1}^{n} \sum_{i=2}^{n} \ddot{d}_i^{(j)}(t) \\
    + k_n x_2 + k_n d_1 + \sum_{j=1}^{n} \ddot{d}_i^{(j)}(t)
\end{bmatrix}
\]

(45)

It can be derived from \( \dot{V}_n \) (45) that the system states will reach the defined sliding surface \( s_n = 0 \) when \( \eta_n > \sup_{t>0} \left\| k_n e_{d1} + \sum_{i=2}^{n} e_{d_i} \right\| \), then the closed-loop system is asymptotically stable.

**Remark 5:** The above proof implies that states of system can be driven to the desired equilibrium point and the control law can force the system states to reach the sliding-mode surface. This is the main reason why the proposed ENDOB-based SMC method is insensitive to matched uncertainties as well as mismatched uncertainties.

Since the discontinuous switching characteristics of sgn function is likely to cause chattering, so traditional sgn function is replaced by the hyperbolic tangent function to reduce chattering. The hyperbolic tangent function is expressed as:

\[
H(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}
\]

(46)

**SIMULATIONS**

To evaluate the effectiveness of the proposed method, two examples are given below.

**Numerical example 1: Second order system**

A second order nonlinear uncertain system [12] is considered as:

\[
\begin{aligned}
    \dot{x}_1 &= x_2 + d_i(t) \\
    \dot{x}_2 &= -2x_1 - x_2 + e^{x_1} + u
\end{aligned}
\]

(47)

In order to show the advantages of the ENDOB-based OSUMC method proposed in this paper to reference [12] and its nominal sliding control counterpart, we will use simulations to compare the performance with them for the system (46). The control parameters of all the control methods are listed in Table 1. Consider the initial states of system (46) as \( x(0) = [1 \ -1] \). Where, \( d_i = 0.5 \) is imposed on the system at 6 sec. The system (46) can be expressed as (9) where \( f(x) = \begin{bmatrix} x_2 & -2x_1 - x_2 + e^{x_1} \end{bmatrix}^T \), \( g_1(x) = [0 \ \ 1]^T \), \( g_2(x) = [1 \ \ 0]^T \).
The performance index $J$ is defined as

$$J = \int_0^\infty x^T \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} x + u_1^T 2u_1 \, dt$$

(48)

The CLF is selected as

$$V(x) = x^T \begin{bmatrix} 4 & 0.2 \\ 0.2 & 2 \end{bmatrix} x$$

(49)

Fig. 1 depicts the two state variables $x_1$ and $x_2$. It can be observed from Figs. 1 that the proposed method and the DOB-SMC proposed by Yang J [12] exhibit a better performance than traditional SMC in case of the same switching gain, and the states responses caused deterioration when the same switching gain is decreased from 16 to 10 of traditional SMC, even failed to reject the disturbances of the system effectively. It is clear that the proposed method fluctuate smaller than reference [12] when the mismatched resistance $d$ is imposed on the system at 6 sec. The proposed method has a better disturbance rejection abilities.

<table>
<thead>
<tr>
<th>Controllers</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>SMC1</td>
<td>$k=25$, $\eta=16$</td>
</tr>
<tr>
<td>SMC2</td>
<td>$k=25$, $\eta=10$</td>
</tr>
<tr>
<td>Reference [12]</td>
<td>$k=25$, $\eta=10$, $l={6,0}$</td>
</tr>
<tr>
<td>OSMC+ENOB</td>
<td>$k=25$, $\eta=10$, $l\in{11}$, $l_1=20$, $l_2=8$</td>
</tr>
</tbody>
</table>

Table 1. Control Parameters for the Numerical Example in Case 1.

Figure 1. System state variables.
Table 2. Control Parameters for the Numerical Example in Case 2.

<table>
<thead>
<tr>
<th>Controllers</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>SMC2</td>
<td>$k=140, \eta=22$</td>
</tr>
<tr>
<td>Reference [12]</td>
<td>$k=140, \eta=22, l=\text{diag {6,6,6}}$</td>
</tr>
<tr>
<td>OSMC+ENDOB</td>
<td>$k=140, \eta=22, l_{11}=5, l_{12}=6$</td>
</tr>
<tr>
<td></td>
<td>$l_{21}=5, l_{22}=6, l_{31}=20, l_{32}=15$</td>
</tr>
</tbody>
</table>

**Numerical Example 2: Third Order System**

Let us consider a third order nonlinear uncertain system given below:

\[
\begin{aligned}
\dot{x}_1 &= x_2 + d_1(t) \\
\dot{x}_2 &= x_3 + d_2(t) \\
\dot{x}_3 &= -2x_2 - x_3 + e^x + u + d_3(t) \\
y &= x_1
\end{aligned}
\]  
(50)

where complex mismatched disturbance is considered:

\[
\begin{aligned}
d_1 &= \begin{cases} 
0 & \text{for } 0 < t < 20 \\
1.5 & \text{for } 20 \leq t < 30 \\
5 + 0.2 \sin(\pi t) & \text{for } 30 \leq t \leq 40
\end{cases} \\
d_2 &= \begin{cases} 
0 & \text{for } 0 < t < 35 \\
0.5 & \text{for } 35 \leq t \leq 40
\end{cases}
\end{aligned}
\]

where matched disturbance $d_3 = 1$ is imposed on the system at 10 sec. The control parameters of all the control methods are listed in Table 2. Consider the initial states of system (49) as $x(0)=[1\ -1\ 1]^T$. The system (49) can be expressed

\[
f(x) = [x_2 \ x_3 \ -2x_2 - x_3 + e^x] \\
eg_1(x) = [0\ 0\ 1]^T \\
eg_2(x) = [1\ 1\ 1]^T
\]

The performance index $J$ is defined as:

\[
J = \int_0^\infty x^T \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix} x + u_i^T 3u_i \, dt
\]  
(51)

The CLF is selected as:

\[
V(x) = x^T \begin{bmatrix} 4 \ 0.17 \ 0.18 \\ 0.17 \ 16 \ 1.9 \\ 0.18 \ 1.9 \ 6 \end{bmatrix} x
\]  
(52)

Fig.2 depicts the three systems state variables, $x_1$, $x_2$, and $x_3$. It can be observed from Figs. 2 that three methods are insensitive to matched uncertainties $d_3$ changes from 0 to 1 at 10s, but when the mismatched resistance $d_2$ is imposed on the system at
20 sec, the states responses caused deterioration of traditional SMC, $d_2$ is imposed on the system at 30 sec, even failed to reject the disturbances of the system effectively. The proposed method and reference [12] stabilize the considered nonlinear uncertain system to the equilibrium state almost at the same rate when mismatched resistance $d_2$ is imposed on the system, but the proposed method can better reject harmonic mismatched disturbance $d_1$ than reference [12].

The reference/estimation uncertainties of the system are shown in Fig. 3. It can be seen from Fig. 3 that the proposed control gives a better estimation of the disturbance and a better performance.

![Figure 2. System state variables.](image1)

![Figure 3. Reference/estimation uncertainties.](image2)
CONCLUSION

In this paper, the matched and mismatched uncertainties rejection control problem have been studied for the second-, third-, and higher-order systems. A novel ENDOB-based OSMC approach has been proposed. The controller guarantee the robustness, maintain the nearly optimal performance of the sliding-mode dynamics, minimize the control input while at the same time reduces chattering. Simulation results reveal the effectiveness of the proposed method.

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REFERENCES