Exact Solutions of Three Nonlinear Systems via Exp-Function Method

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Abstract. In this paper, the exp-function method is used to exactly solve three nonlinear partial differential systems, i.e., the (4+1)-dimensional Fokas equation, the variable-coefficient Burger’s equation and the variable-coefficient Benjamin-Bona-Mahony (BBM) equation. As a result, three exact solutions are obtained. It is shown that the exp-function method is a useful mathematical tool for solving some other high-dimensional nonlinear partial differential equations (PDEs) and variable-coefficient nonlinear PDEs.

Introduction

In 2006, He and Wu [1] proposed a simple and direct method for solving nonlinear PDEs. The existing researches show that the exp-function method and its improvements are available for many nonlinear PDEs, such as those in [2–9]. This is due to the exp-function method possesses a more general ansatz [10]:

$$u = \frac{\sum_{n=-f}^{g} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}, \quad \xi = \sum_{i=1}^{s} k_i x_i + wt,$$  \hspace{1cm} (1)

which is supposed for a given nonlinear PDE with independent variables $t, x_1, x_2, \cdots, x_s$ and dependent variable $u$:

$$F(u, u_t, u_{x_i}, \cdots, u_{x_i}, u_{x_i}, u_{x_i}, u_{x_i}, u_{x_i}, u_{x_i}, \cdots, u_{x_i}) = 0.$$  \hspace{1cm} (2)

where $a_n$ and $b_m$ are undetermined constants, the integers $f$, $p$, $g$ and $q$ can be determined by balancing the highest order linear term with the highest order nonlinear term in Eq. (2). In general, the final solution does not strongly depend on the choices of values of $f$, $p$, $g$, $q$ and usually $f = p = g = q = 1$ is the simplest choice. The present paper is motivated by the desire to extend the exp-function method to the (4+1)-dimensional equation [11]:

$$\frac{\partial^2 u}{\partial t \partial x_i} = \frac{1}{4}(\partial^3 u_{x_i} - \partial^3 u_{x_i} \partial x_i)u - \frac{3}{2} \partial^2 u_{x_i} u_x^2 + \frac{3}{2} \partial^2 u_{x_i} u_{x_i} u_x,$$ \hspace{1cm} (3)

the variable-coefficient Burger’s equation [12]:

$$u_t + \alpha(t)u u_x + \beta(t)u_{xx} = 0,$$  \hspace{1cm} (4)

and the variable-coefficient BBM equation [13]:

$$u_t + \alpha(t)u u_x + \beta(t)u_{xxt} = 0,$$ \hspace{1cm} (5)

where $\alpha(t)$ and $\beta(t)$ are arbitrary functions of $t$. 
**Exact Solutions**

Firstly, we consider the (4+1)-dimensional Eq. (3). Taking the following travelling wave transformation:

$$u = u(\xi), \quad \xi = ax_1 + bx_2 + cy_1 + dy_2 + et,$$

where $a, b, c, d$ and $e$ are undetermined constants, we convert Eq. (3) into an ordinary differential equation (ODE):

$$4eau^{(n)} - a^3bu^{(n)} + ab^3u^{(n)} + 12abu^{(n)^2} + 12abu^{(n)} - 6cdv^{(n)} = 0. \quad (7)$$

Supposing Eq. (7) has a solution of the form:

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{11} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{11} \exp(-\xi)}, \quad (8)$$

where $a_1, a_0, a_{11}, b_1, b_0$ and $b_{11}$ are constants determined later. Substituting Eq. (8) into Eq. (7) and then equating each coefficient of the same order power of $\exp(j\xi)$ ($j = 1, 2, \ldots, 9$) to zero yields a system of nonlinear algebraic equations for $a_1, a_0, a_{11}, b_1, b_0, b_{11}, a, b, c, d$ and $e$:

$$-12a_1^2b_0b_{11}^2 + 12aa_1b_0b_{11} + a_3a_1b_1b_{11}^3 - a_1b_1b_{11}^3 - a_3a_0b_1^4 + aa_0b_1^4 + 6a_1b_1c_{11}d - 6a_1b_1c_{11}^3d - 4aa_0b_1c_{11}^3d + 4aa_0b_1c_{11}^3e = 0, \quad (9)$$

$$12a_1^2b_0b_{11}^2 + 16aa_1b_0^3e - 11a_0a_1b_0^3b_{11}^2 + 11aa_1b_1^3b_{11}^2 - 48aa_0b_1^3b_{11}^3 - 48aa_0b_1^3b_{11}c_{11}d + 16aa_0b_1^3b_{11}c_{11}^3d + 16aa_0b_1^3b_{11}c_{11}^3e = 0, \quad (10)$$

$$76a_1^2b_0b_{11}^3 - 36aa_1b_0b_{11}^2b_{11} + 11a_0a_1b_0b_{11}^3b_{11} + 11aa_1b_1b_{11}b_{11}^3 - 156aa_0a_1b_1b_{11}^3 + 12aa_0b_1^2b_{11}^3 + 12aa_0b_1^2b_{11}c_{11}d + 12aa_0b_1^2b_{11}c_{11}^3d + 12aa_0b_1^2b_{11}c_{11}^3e = 0, \quad (11)$$

$$12aa_0b_1^2b_{11}^3 - 12a_1b_0b_{11}^3 + 12aa_0b_1b_{11}b_{11}^2 + 16aa_1b_0b_{11}^2b_{11} + 12aa_0a_1b_0b_{11}^3b_{11} - 12a_0b_1b_{11}b_{11}^3b_{11}^2 + 12a_0b_1b_{11}c_{11}d + 12a_0b_1b_{11}c_{11}^3d + 12a_0b_1b_{11}c_{11}^3e = 0, \quad (12)$$
+60a_1b_1^2b_1^2cd + 115a_1^3a_1bb_1b_1^2 + 115a_1^3a_1bb_1b_1^2 - 40aa_1b_1^2b_1^2e + 20aa_1b_1^3b_1^2e + 20aa_1b_1^3b_1^2e - 40aa_1b_1^2b_1^2e = 0, \quad (13)

12aa_1a_1bb_1^2 - a^2a_1bb_1^2 + aa_1b_1b_1^2 - 12aa_1^2bb_1^2b_1 - 24aa_1a_1bb_1^2b_1 + a^2a_1bb_1^2b_1

- aa_1b_1^2b_1^2 + 156aa_1a_1bb_1^2b_1 - 11a_1^3a_1bb_1^2b_1 + 11a_1^3bb_1^2b_1 + 11aa_1b_1^2b_1^2 + 96aa_1^2b_1b_1^2 + 108aa_1^2bb_1^2b_1 - 168aa_1a_1bb_1^2b_1 + 58a_1^2bb_1^2b_1^2b_1 - 58aa_1b_1^2b_1^2b_1

- 72aa_1b_1^2b_1^2b_1 - 144aa_1a_1bb_1^2b_1^2 + 47a_1^3a_1bb_1^2b_1^2 + 47aa_1b_1^2b_1^2b_1^2 + 4aa_1b_1^2b_1^2e

- 176aa_1b_1^2b_1^2 + 48aa_1^2bb_1^2b_1 - 176a_1^3bb_1^2b_1^2 + 176aa_1b_1^2b_1^2 - 6a_1b_1^4cd

+ 6aa_1b_1^2cd - 66aa_1b_1^2b_1^2cd - 12a_1b_1^2b_1^2cd + 78aa_1b_1^2b_1^2cd - 24a_1b_1^2b_1^2cd

+ 24aa_1b_1^2b_1^2 - 4aa_1b_1^2b_1^2e + 44aa_1b_1^2b_1^2e + 8aa_1b_1^2b_1^2e - 52aa_1b_1^2b_1^2e

+ 16aa_1b_1^2b_1^2e - 16aa_1b_1^2b_1^2e + 176a_1^3a_1bb_1^2b_1 = 0, \quad (14)

24aa_1^2b_1^2b_1^2 - 36aa_1a_1bb_1^2b_1 + 11a_1^3a_1bb_1^2b_1 - 11aa_1b_1^2b_1^2b_1 + 12aa_1^2bb_1^2b_1 - 16aa_1b_1^2b_1^2e + 24aa_1a_1bb_1^2b_1^2 + 11aa_1b_1^2b_1^2b_1 + 11aa_1b_1^2b_1^2b_1 + 108aa_1a_1bb_1^2b_1^2 + a^2a_1bb_1^2b_1^2

- aa_1b_1^2b_1^2 + 24aa_1a_1bb_1^2b_1^2b_1 - 156aa_1a_1bb_1^2b_1^2b_1 - 77a_1^3a_1bb_1^2b_1^2b_1 + 77aa_1b_1^2b_1^2b_1^2b_1

+ 76a_1^3a_1bb_1^2b_1 - 76aa_1b_1^2b_1^2b_1 - 76aa_1b_1^2b_1^2b_1 - 6a_1b_1^4cd + 6a_1b_1^4cd

- 66aa_1b_1^2cd + 42a_1b_1^2b_1^2cd + 24aa_1b_1^2b_1^2cd - 4aa_1b_1^2b_1^2e + 4aa_1b_1^2b_1^2e - 52aa_1b_1^2b_1^2e

- 28aa_1b_1^2b_1^2e, \quad (15)

12aa_1^2bb_1^2b_1 - 36aa_1a_1bb_1^2b_1 - 11a_1^3a_1bb_1^2b_1 + 11aa_1b_1^2b_1^2b_1 + 24aa_1^2bb_1^2b_1^3 + 48aa_1a_1bb_1^2b_1 + a^2a_1bb_1^2b_1^2 + 11aa_1b_1^2b_1^2b_1 + 16aa_1b_1^2b_1^2b_1 + 16aa_1b_1^2b_1^2b_1 + 108aa_1a_1bb_1^2b_1^2 + a^2a_1bb_1^2b_1^2

- 48aa_1^2bb_1^2b_1^2 + 16aa_1^3a_1bb_1^2b_1 + 16aa_1^3a_1bb_1^2b_1 + 6a_1b_1^4cd - 6a_1b_1^4cd

- 24a_1b_1^2cd + 24a_1b_1^2b_1^2cd - 4aa_1b_1^2b_1^2e + 4aa_1b_1^2b_1^2e + 16aa_1b_1^2b_1^2e - 16aa_1b_1^2b_1^2e = 0, \quad (16)

- 12aa_1^2bb_1^2b_1 + 12aa_1a_1bb_1^2b_1 - aa_1b_1^2b_1^2b_1 - a^2a_1bb_1^2b_1^2 + aa_1b_1^2b_1^2b_1 + 6a_1b_1^4cd - 6a_1b_1^4cd - 4aa_1b_1^2b_1^2e + 4aa_1b_1^2b_1^2e = 0. \quad (17)

Solving the set of algebraic Eqs. (9)-(17) yields:

\[ a_i = \frac{b_1(a_1^3b - ab^3 + 6cd - 4ae)}{12ab}, \quad a_{11} = \frac{b_1(a_1^3b - ab^3 + 6cd - 4ae)}{12ab}, \quad (18) \]

\[ a_0 = -\frac{b_0(5a_1^3b - 5ab^3 - 6cd + 4ae)}{12ab}, \quad b_1 = \frac{b_0^2}{4b_{11}}, \quad (19) \]

And then we obtain an exact solution of Eq. (3):

\[ u = \left(\frac{(a_1^3b - ab^3 + 6cd - 4ae)(b_1e^z + b_{11}e^{-z} - b_0(5a_1^3b - 5ab^3 - 6cd + 4ae))}{12ab\left(\frac{b_0^2}{4b_{11}}e^z + b_1e^{-z}\right)}\right), \quad (20) \]

where \( \xi = ax_1 + bx_2 + cy_1 + dy_2 + et, \ a, \ b, \ c, \ d \) and \( e \) are arbitrary constants.

Secondly, we consider the variable-coefficient Burger’s Eq. (4). We use the transformation:

\[ u = u(\xi), \quad \omega = k(t)x + p(t), \quad (21) \]
where $k(t)$ and $p(t)$ are undetermined functions, then Eq. (4) is converted into an ODE:

$$[K'(t)x + p'(t)]u' + k(t)\alpha(t)uu' + k^2(t)\beta(t)u'' = 0. \tag{22}$$

We suppose that Eq. (22) has a solution of the form:

$$u(\xi) = a_1 e^{\alpha_1 \xi} + a_0 + a_{11} e^{\alpha_{11} \xi},$$

where $a_1$, $a_0$, $a_{11}$, $b_1$, $b_0$ and $b_{11}$ are constants to be determined. Substituting Eq. (23) into Eq. (22) and then equating each coefficient of the same order power of $x \exp(j\xi)$ ($j = 1, 2, 3, 4, 5$) and $\exp(j\xi)$ ($j = 1, 2, 3, 4, 5$) to zero yields a system of nonlinear differential equations for $a_1$, $a_0$, $a_{11}$, $b_1$, $b_0$, $b_{11}$, $k(t)$ and $p(t)$. Solving the set of nonlinear differential equations, we have

$$a_{11} = \frac{a_2^2 b_1 (a_0 b_0 - a_1 b_1)}{(a_0 b_0 - a_1 b_1)^2}, \quad b_1 = \frac{a_1 (a_0 b_0 - a_1 b_1)}{a_0^2}, \quad k(t) = k, \tag{24}$$

$$p(t) = \frac{k(-2a_2^3 b_0 + 3a_2^2 a_1 b_1)}{2(-a_0 b_0 + a_1 b_1)^2} \int \alpha(t) d\xi, \quad \beta(t) = \frac{a_0^2 a_1 b_1 \alpha(t)}{2(a_0 b_0 - a_1 b_1)^2} k^2, \tag{25}$$

and hence obtain an exact solution of Eq. (4):

$$u = \frac{a_1 e^\xi + a_0 + a_2^2 b_1 (a_0 b_0 - 2a_1 b_1)}{a_0^2 (a_0 b_0 - a_1 b_1)^2} e^{-\xi} + b_0 + b_{11} e^{-\xi}, \quad \xi = kx + \frac{k(-2a_2^3 b_0 + 3a_2^2 a_1 b_1)}{2(-a_0 b_0 + a_1 b_1)^2} \int \alpha(t) d\xi. \tag{26}$$

where $a_0$, $a_1$, $b_0$ and $b_{11}$ are arbitrary constants.

Finally, we solve the variable-coefficient BBM Eq. (5). Using the transformation (21) yields an ODE:

$$[K'(t)x + p'(t)]u' + \alpha(t)k(t)uu' + \beta(t)k^2(t)(k'(t)x + p'(t))u'' = 0. \tag{27}$$

We suppose that Eq. (27) has a solution of the form (23). Substituting Eq. (23) into Eq. (27) and then equating each coefficient of the same order power of $x \exp(j\xi)$ ($j = 1, 2, 3, 4, 5$) and $\exp(j\xi)$ ($j = 1, 2, 3, 4, 5$) to zero yields a system of nonlinear differential equations for $a_1$, $a_0$, $a_{11}$, $b_1$, $b_0$, $b_{11}$, $k(t)$ and $p(t)$. Solving the set of nonlinear differential equations, we have

$$a_{11} = \frac{(a_0 + a_1 k^2 \beta(t))^2}{4a_1^2 (1 - 5k^2 \beta(t))^2}, \quad b_1 = \frac{b_{11} (a_0 + a_1 k^2 \beta(t))^2}{4a_1^2 (1 - 5k^2 \beta(t))^2}, \quad b_0 = -\frac{a_0 b_{11} (1 + k^2 \beta(t))}{a_1 (1 + 5k^2 \beta(t))}, \quad k(t) = k, \tag{28}$$

$$\beta(t) = b, \quad p(t) = -\int \frac{a_1 k \alpha(t)}{b_{11} (1 + k^2 \beta(t))} d\xi, \tag{29}$$

and hence obtain an exact solution of Eq. (5):

$$u = \frac{(a_0 + a_1 k^2 \beta(t))^2}{4a_1^2 (1 - 5k^2 \beta(t))^2} e^\xi + a_0 + a_{11} e^{-\xi} - \frac{b_{11} (a_0 + a_1 k^2 \beta(t))^2}{4a_1^2 (1 - 5k^2 \beta(t))} e^\xi + \frac{a_0 b_{11} (1 + k^2 \beta(t))}{a_1 (1 + 5k^2 \beta(t))} + b_{11} e^{-\xi}, \quad \xi = kx - \int \frac{a_1 k \alpha(t)}{b_{11} (1 + k^2 \beta(t))} d\xi. \tag{30}$$

where $a_0$, $a_{11}$ and $b_1$ are arbitrary constants.
Summary
The (4+1)-dimensional Fokas Eq. (3), the variable-coefficient Burger’s Eq. (4) and the variable-coefficient Benjamin-Bona-Mahony (BBM) Eq. (5) have been solved by the exp-function method. As a result, three exact solutions (20), (26) and (30) with free parameters are obtained. This paper shows that the exp-function method is effective for some other high-dimensional nonlinear PDEs and variable-coefficient nonlinear PDEs.

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