Tensor Method to the Weak Solution of the Second-Order System of Ordinary Differential Equations with Boundary Value Problems (II)

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Abstract. In this paper, a tensor method for variational approach is established based on Lax-Milgram theorem. Some new results are obtained for the existence of weak solution of ODE system with homogeneous boundary value problems. By seeking the approximate solutions using Matlab, it is proved that the proposed method is effective to the system of ODE with boundary value problems in practice.

Lemma 3.1
The tensor operator \( a(U \otimes W)_{H\tilde{d}_i(a,b)} \) defined by (4) is a bilinear operator, and the tensor operator \( (F \otimes W)_{H\tilde{d}_i(a,b)} \) defined by (5) is a linear operator.

Proof. According to the definition (4), for all \( \alpha, \beta \in R \) we know
\[
a((\alpha U_1 + \beta U_2) \otimes W)_{H\tilde{d}_i(a,b)} = \int_a^b \left[ A(\alpha U_1 + \beta U_2)'(W^T)' + B(\alpha U_1 + \beta U_2)W^T \right] dx
\]
\[
= \int_a^b \left[ A(\alpha U_1)'(W^T)' + B(\alpha U_1)W^T \right] dx + \int_a^b \left[ A(\beta U_2)'(W^T)' + B(\beta U_2)W^T \right] dx
\]
\[
= \alpha a(U_1 \otimes W)_{H\tilde{d}_i(a,b)} + \beta a(U_2 \otimes W)_{H\tilde{d}_i(a,b)}
\]

And
\[
a(U \otimes (\alpha W_1 + \beta W_2))_{H\tilde{d}_i(a,b)} = \int_a^b \left[ AU'(\alpha W_1 + \beta W_2)' + BU(\alpha W_1 + \beta W_2)W \right] dx
\]
\[
= \alpha a(U \otimes W_1)_{H\tilde{d}_i(a,b)} + \beta a(U \otimes W_2)_{H\tilde{d}_i(a,b)}
\]

According to (5), we get
\[
((\alpha F_1 + \beta F_2) \otimes W)_{H\tilde{d}_i(a,b)} = \int_a^b ((\alpha F_1 + \beta F_2)W^T) dx = \alpha (F_1 \otimes W)_{H\tilde{d}_i(a,b)} + \beta (F_2 \otimes W)_{H\tilde{d}_i(a,b)}
\]
So that the operator \( (F \otimes W)_{H\tilde{d}_i(a,b)} \) is a linear operator. This is complete.

Theorem 3.1 Extend Riesz Representation Theorem

If \( a_y, b_y > 0 \) then there is a linear operator \( K = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} \) such that
\[
a(U \otimes W)_{H\tilde{d}_i(a,b)} = (K(U) \otimes W)_{H\tilde{d}_i(a,b)}
\]
where \( K_i = (1,2,3,4) \) have inverse operator.

Proof. Due to \( a_y, b_y > 0 \), then, the bilinear operators
\[
\int_a^b \left[ a_{1i}u'w_i + b_{1i}uw_i \right] dx, \quad \int_a^b \left[ a_{12i}v'w_i + b_{12i}vw_i \right] dx,
\]
\[
\int_a^b \left[ a_{2i}u'w_i + b_{2i}uw_i \right] dx, \quad \int_a^b \left[ a_{21i}v'w_i + b_{21i}vw_i \right] dx, \quad (i = 1,2)
\]
are systemical, bounded and elliptic. According to Lax-Milgram theorem, then there exists linear inverse operators $K_1, K_2, K_3$ and $K_4$ such that

$$
\int_0^b \left[ a_{11}u'w' + b_{11}uw \right] dx = (K_1(u), w)_{H^1_0(a,b)}, \\
\int_0^b \left[ a_{21}v'w' + b_{21}vw \right] dx = (K_2(v), w)_{H^1_0(a,b)}, \\
\int_0^b \left[ a_{22}u'w' + b_{22}uw \right] dx = (K_3(u), w)_{H^1_0(a,b)}, \\
\int_0^b \left[ a_{32}v'w' + b_{32}vw \right] dx = (K_4(v), w)_{H^1_0(a,b)},
$$

(7)

where $(\bullet, \bullet)$ is the inner product in $H^1_0(a,b)$.

Therefore

$$
a(U \otimes W)_{H^2_0(a,b)} = \begin{pmatrix} (K_1(u), w_1) + (K_2(v), w_1) \\ (K_3(u), w_1) + (K_4(v), w_1) \end{pmatrix} = \begin{pmatrix} (K_1(u) + K_2(v)) \\ (K_3(u) + K_4(v)) \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_{H^2_0(a,b)}
$$

(8)

Noting that

$$
\begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = (K(U) \otimes W)_{H^2_0(a,b)}
$$

Then, there exists an operator

$$
K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}
$$

such that

$$a(U \otimes W)_{H^2_0(a,b)} = (K(U) \otimes W)_{H^2_0(a,b)}.$$

(9)

**Lemma 3.1**

The operator $K$; defined by (9) is a linear operator.

**Proof:** Let $U_1 = (u_1, v_1)^T$ and $U_2 = (u_2, v_2)^T$. Since $K_i = (1, 2, 3, 4)$ are linear operators, then, for all $\alpha, \beta \in \mathbb{R}$ we have

$$
K(\alpha U_1 + \beta U_2) = \begin{pmatrix} \alpha K_1(u) + \beta K_1(u_2) + \alpha K_2(v) + \beta K_2(v_2) \\ \alpha K_3(u) + \beta K_3(u_2) + \alpha K_4(v) + \beta K_4(v_2) \end{pmatrix} = \alpha K(U_1) + \beta K(U_2)
$$

Hence, the operator $K$ is a linear operator. This is complete.

**Definition 3.2 Inverse operator**

The operator $K$ is called inverse, if there exists $K^{-1}$ such that

$$K^{-1}(K(U)) = K(K^{-1}(U)) = IU$$

i.e., $K^{-1}K = KK^{-1} = I$, where $I$ is the identity operator.

According to (7), we suppose that $Ki$ are inverse operators. If $K$ has left inverse $K^{-1}_L$, letting

$$
K^{-1}_L K(U) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}
$$

then
\[
\begin{aligned}
(\alpha K_1 + \beta K_3)(u) + (\alpha K_2 + \beta K_4) &= u \\
(\gamma K_1 + \delta K_3)(u) + (\gamma K_2 + \delta K_4) &= v
\end{aligned}
\]

i.e.,
\[
\begin{align*}
\alpha K_1 + \beta K_3 &= I, & \alpha K_2 + \beta K_4 &= 0 \\
\gamma K_1 + \delta K_3 &= 0, & \gamma K_2 + \delta K_4 &= I
\end{align*}
\]

where \(I\) is the identity operator and \(0\) is the zero operator.

By right multiplying \(K_3^{-1}\), we get
\[
\begin{align*}
\alpha K_1 K_3^{-1} + \beta K_2 K_3^{-1} &= K_3^{-1} \\
\alpha K_2 K_3^{-1} + \beta K_4 K_3^{-1} &= 0
\end{align*}
\]

i.e.,
\[
\alpha(K_1 K_3^{-1} - K_2 K_4^{-1}) = K_3^{-1}
\]

If \(K_1 K_3^{-1} - K_2 K_4^{-1} \neq 0\), which is inverse, then
\[
\alpha = K_3^{-1}(K_1 K_3^{-1} - K_2 K_4^{-1})^{-1} = \left[(K_1 K_3^{-1} - K_2 K_4^{-1}) K_3 \right]^{-1} = (K_1 - K_2 K_4^{-1} K_3)^{-1}
\]

In a similar way, by right multiplying \(K_2^{-1}, K_3^{-1}\), and \(K_4^{-1}\), when \(K_1 K_3^{-1} - K_2 K_4^{-1} \neq 0\),
\[
K_2 K_4^{-1} - K_1 K_3^{-1} \neq 0, K_3 K_2^{-1} - K_4 K_1^{-1} \neq 0,
\]
we obtain
\[
\begin{align*}
\beta &= K_3^{-1}(K_3 K_1^{-1} - K_4 K_2^{-1})^{-1} = \left[(K_3 K_1^{-1} - K_4 K_2^{-1}) K_1 \right]^{-1} = (K_3 - K_4 K_2^{-1} K_1)^{-1} \\
\gamma &= K_4^{-1}(K_2 K_4^{-1} - K_3 K_5^{-1})^{-1} = \left[(K_2 K_4^{-1} - K_3 K_5^{-1}) K_4 \right]^{-1} = (K_2 - K_3 K_1^{-1} K_4)^{-1} \\
\delta &= K_1^{-1}(K_2 K_1^{-1} - K_3 K_1^{-1})^{-1} = \left[(K_2 K_1^{-1} - K_3 K_1^{-1}) K_2 \right]^{-1} = (K_2 - K_3 K_1^{-1} K_4)^{-1}
\end{align*}
\]

If \(K\) has right inverse \(K_R^{-1}\), letting
\[
KK_R^{-1}(U) = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}
\]
then
\[
\begin{align*}
(K_1 \alpha_i + K_2 \gamma_i)(u) + (K_4 \beta_i + K_2 \delta_i) &= u \\
(K_3 \alpha_i + K_4 \gamma_i)(u) + (K_3 \beta_i + K_4 \delta_i) &= v
\end{align*}
\]

So that
\[
\begin{align*}
K_1 \alpha_i + K_2 \gamma_i &= I, & K_4 \beta_i + K_2 \delta_i &= 0, \\
K_3 \alpha_i + K_4 \gamma_i &= 0, & K_3 \beta_i + K_4 \delta_i &= I,
\end{align*}
\]

where \(I\) is the identity operator and \(0\) is the zero operator.

If \(K_1 K_3^{-1} - K_2 K_4^{-1} \neq 0, K_4 K_3^{-1} - K_2 K_1^{-1} \neq 0, K_1 K_2^{-1} - K_3 K_4^{-1} \neq 0, K_1 K_4^{-1} - K_2 K_3^{-1} \neq 0\), by left multiplying \(K_1^{-1}, K_2^{-1}, K_3^{-1}, K_4^{-1}\), we obtain
\[
\begin{align*}
\alpha_i &= (K_2^{-1} K_1^{-1} - K_4^{-1} K_3^{-1})^{-1} K_2^{-1} = (K_1 - K_2 K_4^{-1} K_3)^{-1} = \alpha \\
\beta_i &= (K_2^{-1} K_3^{-1} - K_2^{-1} K_1^{-1})^{-1} K_4^{-1} = (K_3 - K_4 K_2^{-1} K_1)^{-1} = \beta \\
\gamma_i &= (K_1^{-1} K_2^{-1} - K_3^{-1} K_4^{-1})^{-1} K_1^{-1} = (K_2 - K_3 K_4^{-1} K_1)^{-1} = \gamma \\
\delta_i &= (K_3^{-1} K_4^{-1} - K_1^{-1} K_2^{-1})^{-1} K_3^{-1} = (K_4 - K_1 K_2^{-1} K_4)^{-1} = \delta
\end{align*}
\]

Hence, we get
According to the above, we have the following theorem.

Theorem 3.2
If the operator \( K \), defined by (9), has an inverse operator \( K^{-1} \), then there is a unique weak solution \( U \) such that

\[
(\mathbf{u}, \mathbf{w}) = (K(\mathbf{u}) \otimes \mathbf{w})_{H_{0}^{1}(\Omega)}, \quad \forall \mathbf{w} \in H_{0}^{1}(\Omega)
\]

Proof: If the operator \( K \), defined by (9), has an inverse operator \( K^{-1} \), then there is a weak solution \( U \) such that

\[
a(U \otimes W)_{H_{0}^{1}(\Omega)} = (K(U) \otimes W)_{H_{0}^{1}(\Omega)}
\]

If there exists other \( K^{-1} \) such that

\[
a(U \otimes W)_{H_{0}^{1}(\Omega)} = (K_{i}(U) \otimes W)_{H_{0}^{1}(\Omega)},
\]

then

\[
(K(U) \otimes W)_{H_{0}^{1}(\Omega)} = (K_{i}(U) \otimes W)_{H_{0}^{1}(\Omega)}
\]

It means that

\[
((K(U) - K_{i}(U)) \otimes W)_{H_{0}^{1}(\Omega)} = 0.
\]

By lemma (1), we know \( K(U) - K_{i}(U) = 0 \), i.e., \( K(U) = K_{i}(U) \). This is complete.

Theorem 3.3 Existence of weak solutions
Let \( a_{ij} > 0 \) and \( b_{ij} > 0 \).

(1) If \( \frac{a_{11}}{a_{12}} = \frac{a_{12}}{a_{22}} \) or \( \frac{b_{11}}{b_{12}} = \frac{b_{12}}{b_{22}} \), and functions \( f_{i} \) (\( i = 1; 2 \)) are continuous, then there exists a unique weak solution of system of ODE (3);

(2) If \( \frac{a_{11}}{a_{12}} = \frac{a_{12}}{a_{22}} \neq \frac{b_{11}}{b_{12}} = \frac{b_{12}}{b_{22}} \) and functions \( f_{i} \) (\( i = 1; 2 \)) are continuous, and \( f_{1} = f_{2} \), then there exists several weak solutions of system of ODE (3);

(3) If \( \frac{a_{11}}{a_{12}} = \frac{a_{12}}{a_{22}} \neq \frac{b_{11}}{b_{12}} = \frac{b_{12}}{b_{22}} \) and \( f_{1} \neq f_{2} \), then there not exists weak solution of system of ODE (3).

Proof: We only prove result (1). Since \( \frac{a_{11}}{a_{12}} = \frac{a_{12}}{a_{22}} \) or \( \frac{b_{11}}{b_{12}} = \frac{b_{12}}{b_{22}} \), then, for all \( C_{i} \) (\( i = 1; 2; 3; 4 \);

\[
K_{1} \neq C_{1}K_{2}, K_{1} \neq C_{2}K_{3}, K_{1} \neq C_{3}K_{4}, K - 2 \neq C_{4}K_{4}.
\]

It means that

\[
\begin{cases}
K_{1}K_{3}^{-1} \neq K_{1}K_{4}^{-1}, K_{3}K_{1}^{-1} \neq K_{4}K_{2}^{-1} \\
K_{2}^{-1}K_{1} \neq K_{4}^{-1}K_{3}, K_{1}^{-1}K_{2} \neq K_{1}^{-1}K_{4}
\end{cases}
\]

According to Lax-Milgram theorem, there exists a unique \( K \) such that

\[
a(U \otimes W)_{H_{0}^{1}(\Omega,a,b)} = (K(U) \otimes W)_{H_{0}^{1}(\Omega,a,b)}, \forall W \in H_{0}^{1}(\Omega,a,b)
\]

At the same time, for all linear bounded function \( F = (f_{1}, f_{2})^{T} \), \( f_{i} \) are linear bounded functions, there exists a unique \( K \) such that

\[
(K(U) \otimes W)_{H_{0}^{1}(\Omega,a,b)} = (F \otimes W)_{H_{0}^{1}(\Omega,a,b)}, \forall W \in H_{0}^{1}(\Omega,a,b)
\]

where \( U = K^{-1}F \).

Hence, there exists a unique \( U = K^{-1}F \in H_{0}^{1}(\Omega,a,b) \) such that

\[
a(U \otimes W)_{H_{0}^{1}(\Omega,a,b)} = (F \otimes W)_{H_{0}^{1}(\Omega,a,b)}.
\]

This is complete.

According to matrix theory, \( K \) is inverse, we can define
$|K| = \begin{vmatrix} K_1 & K_2 \\ K_3 & K_4 \end{vmatrix} = K_1 K_4 - K_2 K_3 \neq 0$.

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References


